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A NONCOMMUTATIVE ANALOG OF THE COHEN THEOREM

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By using weakly primary right ideals, we prove an analog of the Cohen theorem for rings of principal right ideals.

Cohen [1] proved that if an arbitrary prime ideal of a commutative ring with identity is principal (finitely generated), then all ideals are principal (finitely generated). In [2–4], these results were generalized to the cases of duo-rings, matrix local rings, and Noetherian rings. In the present paper, we prove an analog of the Cohen theorem for rings of principal right ideals [5–7].

All rings considered below are associative rings with nonzero identity. We also assume that ideals are two-sided. A right ideal P of a ring R is called a weakly primary right ideal if the condition $(a+P)R(b+P) \subseteq P$, where $a, b \in R$, always implies that $a \in P$ or $b \in P$ [8]. A right ideal P of a ring R is called a primary right ideal if the condition $aRb \subseteq P$, where $a, b \in R$, always yields $a \in P$ or $b \in P$ [4]. An element $a \in R$, $a \neq 0$, is called a *duo-element* if $aR = Ra$.

Denote by S the set of all right ideals of a ring R which are not principal right ideals. We call them *nonprincipal right ideals.*

Definition 1. *A nonprincipal right ideal $I \in S$ is called a maximally nonprincipal right ideal if any right ideal in R that contains I is properly a principal right ideal.*

The set S regarded as a partially ordered set with respect to inclusion is an induced set. By virtue of the Zorn lemma, the following statement is true:

Proposition 1. *Any nonprincipal right ideal is contained in at least one maximally nonprincipal right ideal.*

Proposition 2. *Let R be a ring with a single maximally nonprincipal right ideal N. Then N is a primary right ideal.*

Proof. Assume that R contains elements $a \notin N$ and $b \notin N$ such that $aRb \subseteq N$. Since N is maximal, the set S contains an element $c \in R$ such that $N + aR = cR$. Consider the right ideal $J = \{x \mid cx \in N\}$. It is clear that $cb \in N$, i.e., $b \in J$. By virtue of Proposition 1 and the conditions imposed on the ring R, this implies that $J = dR$ is a principal right ideal because $J \not\subset N$. Taking into account that $N + aR = cR$, we get $n + ar = c$, where $n \in N$ and $r \in R$. Since $N \subset cR$ and $aR \subset cR$, for any $m \in N$, there exist elements $l, k \in R$ such that $m = cl$ and $a = ck$. Hence, $m = cl = nl + arl = nl + ckrl$. Further, $c(krl) = m - nl \in N$ and, therefore, $krl = dt$ for some $t \in R$. Consequently, $m = nl + cdt$. Since $nl \in N$, we have $nl = cs$, where, obviously, $s \in J$, i.e., $s = dy$ $(y \in R)$. Hence, $N \subseteq cdR$. Since $1 \in R$, we have $cd \in N$, i.e., $cdR \subset N$. Thus, $N = cdR$ is a principal right ideal of the ring R, which contradicts the choice of the right ideal N. The proposition is proved.

Taking into account that a right chain ring has at most one maximally nonprincipal right ideal, we arrive at the following statement:

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Theorem 1. *A right chain ring is a ring of principal right ideals if and only if every primary right ideal is a principal right ideal.*

Theorem 2. *If every weakly primary right ideal of a ring is a principal right ideal, then any right ideal is a principal right ideal.*

Proof. Assume the contrary. Then the ring R contains maximally nonprincipal right ideals. Let N be an arbitrary nonprincipal right ideal of the ring R. Then, by the choice of R, N is not a weakly primary right ideal, i.e., R contains elements $a, b \notin N$ such that $(a+N)R(b+N) \subset N$. By virtue of the choice of N, we have $n + aR = cR$, $c \in R$. Consider the right ideal $J = \{x \mid cx \in N\}$. Since $aRN \subset N$, we have $cRN \subset N$ and, hence, $N \subset J$. Moreover, $b \in J$ and $b \notin N$. Consequently, $J = dR$ and, in particular, $cd \in N$. Since $N \subset cR$, for any $m \in N$, there exists an element $x \in R$ such that $m = cx$. This obviously implies that $x \in J$. Consequently, $x = dy$, $y \in R$, and $m = cdy$. In other words, $N \subseteq cdR$. Since $cd \in N$, we have $cdR \subseteq N$. Therefore, $N = cdR$, which contradicts the choice of the right ideal N. The theorem is proved.

Definition 2. *A right ideal P of a ring R is called an almost prime right ideal if the condition $ab \in P$, where b is a duo-element of R, always implies that $a \in P$ or $b \in P$. An ideal of R satisfying these conditions is called an almost prime ideal of R.*

Proposition 3. *A maximum two-sided ideal of R is an almost prime ideal.*

Proof. Let M be a maximum two-sided ideal of R (i.e., an ideal such that, for any ideal N, the condition $M \subset N$ with $N \neq M$ always yields $N = R$). Let M be not almost prime. Then there exist an element $a \in R \setminus M$ and a duo-element $b \in R \setminus M$ such that $ab \in M$. Since M is maximal, we have $M + bR = R$. Hence, there exist elements $m \in M$ and $r \in R$ such that $m + br = 1$. This yields $am + abr = a \in M$. Thus, we arrive at a contradiction with the choice of an element a. This contradiction proves the proposition.

Since an arbitrary primary right ideal is an almost prime right ideal, the following statement is true:

Proposition 4. *A maximal right ideal of a ring is an almost prime right ideal.*

Proposition 5. *In a domain where any ideal is a principal right ideal, any nonzero almost prime ideal is a maximal ideal.*

Proof. Let P be an almost prime nonzero proper ideal which is not maximal. Then there exists a maximal ideal M of a ring R such that $P \subset M \neq R$. By the definition of R, we have $P = pR$ and $M = mR$. This implies that $p = rm$ and $m \notin P$. Therefore, $r \in P$, i.e., $r = ps$, $s \in R$. Consequently, $p = psm$ and $m = 1$, which is impossible. The proposition is proved.

If a ring has no nontrivial (other than identity) duo-elements, then, clearly, every right ideal is an almost prime right ideal.

Theorem 3. *For the domain R, the following conditions are equivalent:*

(i) any ideal in R is a principal ideal (both right and left);

(ii) every primary ideal in R is a principal ideal (both right and left);

(iii) every almost prime right ideal in R is a principal ideal (both right and left).

Proof. By virtue of the reasoning presented above, it is sufficient to prove the implication (ii) $\Rightarrow$ (i).

Assume that R contains nonprincipal two-sided ideals. By the Zorn lemma, contains there is an ideal N in R which is maximal among such ideals. By the condition, N is not a primary ideal, i.e., R contains elements $a \in R \setminus N$ and $b \in R \setminus N$ such that $aRb \subseteq N$. By definition, $N + RaR = cR = Rc$ is a principal ideal of R. It is clear that $J = \{x \mid cx \in N\}$ is an ideal of R. By the definition of N, we have $N \subset J$ and, moreover, $N \neq J$ because $b \in J$. This yields $J = dR = Rd$. Since $N \subset cR$ and $RaR \subset cR$, we conclude that $a = cl$, $l \in R$, and any element $m \in N$ can be represented in the form $m = ct, t \in R$. Since $N + RaR = cR$, there exist elements $n \in N$, $r_i, s_i \in R$, $i = 1, 2, \dots, k$, such that

$$n + \sum_{i=1}^{k} r_i a s_i = c.$$

Hence,

$$m = nt + \sum_{i=1}^{k} r_i c l s_i t = nt + cr,$$

where $r \in J$, i.e., $r = ds$, $s \in R$. Then $m = nt + cds$. Since $nt = cz$, we have $z \in J$, $z = dx$, $x \in R$. Then $nt = cdz$. This implies that $N \subset cdR$. Taking into account that $d \in J$ and $1 \in R$, we get $cdR \subseteq N$. Thus, $N = cdR = Rcd$.

Thus, we arrived at a contradiction with the choice of N. The theorem is proved.

The statement below shows that, under certain restrictions imposed on a ring, it is sufficient to demand that maximal ideals be principal.

Theorem 4. *In order that an arbitrary element be a principal ideal (both right and left) in a domain R, it is sufficient that the following conditions be satisfied:*

(i) any maximal ideal of the domain R is a principal ideal (both right and left);

(ii) any maximal ideal N of the domain R satisfies the condition

$$\bigcap_{k=1}^{\infty} M^k = (0).$$

Proof. Assume that R contains ideals that are not principal (both right and left). Then R contains a maximal ideal N with the same property. Let M be a maximal ideal such that $N \subset M$ and $M = mR$. Then, for any nonzero element $n \in N$, we have $n = a_1 m$. Since $m \notin N$, by virtue of Proposition 5, we have $a_1 \in N$ and, therefore, $a_1 = a_2 m$, where $a_2 \in N$, and so on. This implies that $n \in \bigcap_{k=1}^{\infty} M^k$, i.e., $n = 0$, which contradicts the choice of n. The theorem is proved.

Theorem 5. *A ring is a right Noetherian ring if and only if every almost prime right ideal is a finitely generated right ideal.*

Proof. We prove this theorem by contradiction. Assume that R contains a right ideal which is not finitely generated. It is easy to see that the set of such right ideals is an induced set with respect to the order of inclusion.

By the Zorn lemma, there exists a right ideal N which is maximal in the set of such ideals. By assumption, N is not an almost prime right ideal, i.e., R contains an element $a \in R \setminus N$ and a duo-element $b \in R \setminus N$ such that $ab \in N$. Then $N + bR$ is a finitely generated right ideal, i.e.,

$$N + bR = \sum_{i=1}^{k} (n_i + br_i)R.$$

It is clear that the set $J = \{x \mid xb \in N\}$ is a right ideal of R_i and $N \subset J$, $N \neq J$. This implies that $J = \sum_{i=1}^{k} s_i R$. Since $N \subset N + bR$, for any $m \in N$, we have

$$m = \sum_{i=1}^{k} (n_i + br_i)x_i = \sum_{i=1}^{k} n_i x_i + b \sum_{i=1}^{k} r_i x_i.$$

Therefore,

$$m - \sum_{i=1}^{k} n_i x_i = cb \in N, \quad b \sum_{i=1}^{k} r_i x_i \in N,$$

i.e.,

$$m = \sum_{i=1}^{k} n_i x_i + \sum_{i=1}^{k} s_i b y_i.$$

This yields

$$N \subset \sum_{i=1}^{k} n_i R + \sum_{i=1}^{k} s_i bR.$$

Since $s_j \in J$, $j = 1, 2, \ldots, t$, we conclude that

$$\sum_{i=1}^{k} n_i R + \sum_{i=1}^{k} s_i bR = N$$

is a finitely generated right ideal. Thus, we arrived at a contradiction with the choice of N. The theorem is proved.

By analogy, we establish the following result:

Theorem 6. *If every almost prime ideal in a ring is a finitely generated right ideal, then any ideal of the ring is a finitely generated right ideal.*

Below, we present two corollaries of the results obtained above.

Theorem 7. *A ring of finitely generated principal right ideals is a ring in which any ideal is a principal right ideal if and only if any almost prime ideal is a principal right ideal.*

Theorem 8. *A ring of finitely generated principal right ideals is a ring of principal right ideals if and only if any almost prime ideal is a principal right ideal.*

REFERENCES

1. J. S. Cohen, "Commutative rings with restricted minimum conditions," *Duke Math. J.*, **17**, 24–42 (1950).
2. R. Chandran, "On two analogues of Cohen's theorem," *Indiana J. Pur. Appl. Math.*, **8**, No. 1, 54–59 (1977).
3. M. I. Dubrovin, "On rings of principal right ideals," *Izv. Vyssh. Uchebn. Zaved., Ser. Mat.*, No. 2, 30–37 (1981).
4. G. Michler, "Prime right ideals and right Noetherian rings," in: *Proceedings of the Conference on Ring Theory*, Academic Press, New York–London (1972), pp. 251–253.
5. B. V. Zabavskii, "On maximal elements of the set of nonprincipal ideals of a commutative Bezout domain," in: *17 th All-Union Algebraic Conference* [in Russian], Vol. 2, Minsk (1983), p. 75.
6. B. V. Zabavskii, "On one generalization of the Cohen theorem," in: *19 th All-Union Algebraic Conference* [in Russian], Vol. 1, Lvov (1987), p. 99.
7. B. V. Zabavskii, "On the Cohen theorem for noncommutative rings," in: *II Mathematical Conference of Young Scientists* [in Russian], Vol. 1, Institute of Applied Problems in Mechanics and Mathematics, Ukrainian Academy of Sciences, Lvov (1985), pp. 64–65.
8. A. P. Van der Walt, "Weakly prime one-sided ideals," *J. Austr. Math. Soc.*, **38**, 84–91 (1985).