Chapter 1
Bass stable range

1.1 Stable range conditions

Definition 1.1. Let R be a ring. If $(a_1,\dots,a_n),(b_1,\dots,b_n)\in R^n$ and $b_1a_1+\dots+b_na_n=1$, we say that $(a_1,\dots,a_n)$ is a *left unimodular row* over R and $(b_1,\dots,b_n)$ is a *right unimodular row* over R. Also the number n we will call the length of unimodular row, and the set of left (right) unimodular rows of length n we will denote $\mathrm{Um}_n^l(R)$ ($\mathrm{Um}_n^r(R)$). The superscript will be dropped when the row is both left and right unimodular.

Clearly one can say that a nonzero row $(a_1,\dots,a_n)$ is left (right) unimodular if and only if $(a_1,\dots,a_n)$ generates a trivial left (right) ideal.

The following notion is crucial in current work. First it was introduced by Bass [3] for the purposes of stabilizing linear groups over rings and since that time it is extensively used in algebraic K-theory. We give it in some modified form that in fact is equivalent to the original one as we will see in the theorem below.

Definition 1.2. [3] We say that a positive integer n is in the *stable range* of a ring R if for every right unimodular row $(b_1,\dots,b_{n+1})\in R^{n+1}$ there exists $(a_1,\dots,a_{n+1})\in R^n$ such that

$$b_1a_1+\dots+b_{n+1}a_{n+1}=1$$

and $(a_1,\dots,a_n)$ is a left unimodular row. In such case we also say that R is a stable range n ring, or R satisfies $n-$stable range condition or just write that $\mathrm{st.r.}(R)=n$. If there is no such n then we say that the stable range of R is infinite and write this fact as $\mathrm{st.r.}(R)=\infty$.

The notable moment in the definition above is that one can "drop" the element a_{n+1} and use the left linear combination of $a_1,\dots,a_n$ instead of it.

Remark 1.1. As it is notified in the definition the integer n is only "in" the stable range of R and it is not accidentally. If one consider a right unimodular row $(b_1,\ldots,b_{n+m+1})$ $inR^(n+m+1)$ of length $n+m$ (where $m \geq 1$) then there is a left unimodular row $(t_1,\ldots,t_{n+m+1}) \in R^{n+m+1}$ such that

$$b_1t_1 + \ldots + b_nt_n + b_{n+1}t_{n+1} + \ldots + b_{n+m+1}t_{n+m+1} = 1.$$

Therefore $(b_1,\ldots,b_n,b_{n+1}t_{n+1} + \ldots + b_{n+m+1}t_{n+m+1}) \in R^{n+1}$ is also right unimodular and if st.r.$(R) = n$ then there is $(a_1,\ldots,a_{n+1}) \in R^n$ such that

$$b_1a_1 + \ldots + b_na_n + (b_{n+1}t_{n+1} + \ldots + b_{n+m+1}t_{n+m+1})a_{n+1} = 1,$$

where $(a_1,\ldots,a_n)$ is a left unimodular row. Hence the row

$$(a_1,\ldots,a_n,t_{n+1}a_{n+1},\ldots,t_{n+m+1}a_{n+1}) \in R^{n+m+1}$$

satisfies the stable range $n+m$ definition above as first $n+m$ coordinates of the latter row are left unimodular.

Hence it is naturally to speak that the stable range of R is a least possible positive integer n that satisfies the stable range condition for R, but keep in mind that in fact any bigger m also satisfies this condition, i.e. st.r.$(R) = \{n, n+1, \ldots\}$.

If we write down the definition elementwise we obtain the following simple result.

Proposition 1.1. *For a ring R the following conditions are equivalent:*

(i) for every right unimodular row $(b_1,\ldots,b_{n+1}) \in R$ *there exists* $(a_1,\ldots,a_{n+1}) \in R^{n+1}$ *such that*

$$b_1a_1 + \ldots + b_{n+1}a_{n+1} = 1,\ Ra_1 + \ldots Ra_n = R;$$

(ii) for every right unimodular row $(u_1,\ldots,u_{n+1}) \in R$ *there exists* $(x_1,\ldots,x_n) \in R^{n+1}$ *such that*

$$(u_1,\ldots,u_n) + u_{n+1}(x_1,\ldots,x_n) = (u_1 + u_{n+1}x_1,\ldots,u_n + u_{n+1}x_n)$$

is right unimodular.

Proof. To prove (ii) suppose that (i) holds. Then there are $t_1,\ldots,t_n \in R$ such that $a_{n+1} = t_1a_1 + \ldots + t_na_n$. Hence

$$1=\sum_{k=1}^{n+1} b_k a_k=\sum_{k=1}^{n} b_k a_k+b_{n+1}\left(\sum_{k=1}^{n} t_k a_k\right)=\sum_{k=1}^{n}(b_k+b_{n+1}t_k)a_k$$

so $(b_1,\ldots,b_n)+b_{n+1}(t_1,\ldots,t_n)$ is right unimodular.

Conversely if $(b_1,\ldots,b_{n+1})$ is right unimodular then by (ii) there exist $t_1,\ldots,t_n$ $(b_1,\ldots,b_n)+b_{n+1}(t_1,\ldots,t_n)$ is right unimodular. Hence there are $a_1,\ldots,a_n\in R$ such that

$$1=(b_1+b_{n+1}t_1)a_1+\ldots+(b_n+b_{n+1}t_n)a_n$$

and so $(a_1,\ldots,a_n)$ is left unimodular. Moreover, the latter equality implies

$$b_1a_1+\ldots+b_na_n+b_{n+1}\left(\sum_{k=1}^{n} t_k a_k\right)=1$$

and we obtain (i) for $a_{n+1}=\sum_{k=1}^{n} t_k a_k$. □

Since the stable range definition is given for the case of right unimodular rows then it is naturally to introduce this notion as "right stable range" emphasising on the type of unimodular rows we use, and define the "left stable range" in similar manner. However due to the results of Vasserstein [68, 69] and Warfield [71] this notions coincide and we simply speak about the "stable range" of a ring R.

Theorem 1.1. ***(Stable range left-right symmetry)*** *Let R be a ring. Then* $\mathrm{st.r.}(R)=\mathrm{st.r.}(R^{op})$, *where R^{op} means the opposite ring to R.*

Proof. First we prove that $\mathrm{st.r.}(R)\le\mathrm{st.r.}(R^{op})$. Suppose that $\mathrm{st.r.}(R^{op})=n$ and $(b_1,\ldots,b_{n+1})\in R^{n+1}$ is a right unimodular row. Then there are $a_1,\ldots,a_{n+1}\in R$ such that

$$b^T a+b_{n+1}a_{n+1}=1,$$

where $a=(a_1,\ldots,a_n)^T,\ b=(b_1,\ldots,b_n)^T$. Then

$$Ra_1+\ldots+Ra_n+Rb_{n+1}a_{n+1}=R\Rightarrow\sum_{i=1}^{n}R(a_i+v_ib_{n+1}a_{n+1})=R$$

for some $v=(v_1,\ldots v_n)^T\in R^n$ by $\mathrm{st.r.}(R^{op})=n$. Hence one can find $c=(c_1,\ldots c_n)^T\in R^n$ such that

$$-b_{n+1}=c^T(a+va_{n+1}b_{n+1}).$$

Consider the invertible matrix

$$B = \begin{pmatrix} 1 & b^T & b_{n+1} \\ 0 & I_n & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where I_n denotes the identity $n \times n$ matrix. Since every row in invertible matrix is right unimodular then and we are going to obtain the only nonzero elements $b_1 + u_1 b_{n+1}, \ldots, b_n + u_n b_{n+1}$ in the first row of B using elementary transformations of rows and columns of B. First we destroy the identity in the position $(1,1)$:

$$C = B \begin{pmatrix} 1 & 0 & 0 \\ -a & I_n & 0 \\ -a_{n+1} & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & b^T & b_{n+1} \\ -a & I_n & 0 \\ -a_{n+1} & 0 & 1 \end{pmatrix}.$$

Then we can make 0 in the position $(3,1)$ using the above equality for $-b_{n+1}$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & I_n & 0 \\ 0 & c^T & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -v & I_n & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_n & vb_{n+1} \\ 0 & 0 & 1 \end{pmatrix} C = \begin{pmatrix} 0 & b^T & b_{n+1} \\ -a - vb_{n+1}a_{n+1} & I_n - vb^T & 0 \\ 0 & c^T(I - vb^T) & 1 \end{pmatrix}.$$

To simplify the following operations let $u^T = -c^T(I - vb^T)$, and $p = -a - vb_{n+1}a_{n+1}$. Thereafter we can make zeros at $(3,2)$ and then at $(1,3)$:

$$\begin{pmatrix} 1 & 0 & -b_{n+1} \\ 0 & I_n & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & b^T & b_{n+1} \\ p & I_n - vb^T & 0 \\ 0 & -u^T & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_n & 0 \\ 0 & u^T & 1 \end{pmatrix} = \begin{pmatrix} 0 & b^T + b_{n+1}u^T & 0 \\ p & I_n - vb^T & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and as the first row is right unimodular we obtain that

$$(b_1 + b_{n+1}u_1)R + \ldots + (b_n + b_{n+1}u_n)R = R.$$

Hence $\text{st.r.}(R) \leq n = \text{st.r.}(R^{op})$ and we are done as $(R^{op})^{op} = R$. □

Now we will prove two results that state that stable range of quotient-ring R/I cannot exceed the stable range of R, and they coincide when I is radical ideal.

Proposition 1.2. *If I is a two-sided ideal of R then* $\text{st.r.}(R/I) \leq \text{st.r.}(R)$.

Proof. Suppose that $\text{st.r.}(R)$ equals n and $\overline{R} = R/I$. Let $\overline{b} \in \text{Um}^r_{n+1}(\overline{R})$. Then there is some $\overline{a} \in \text{Um}^l_{n+1}(\overline{R})$ such that $c = b^T a - 1 \in I$. Hence

$$b_1 R + b_2 R + \ldots + b_n R + (b_{n+1}a_{n+1} - c)R = R.$$

As st.r.$(R) = n$ then there are $x_1, \dots, x_n \in R$ such that

$$(b_1 + (b_{n+1}a_{n+1} - c)x_1, \dots, b_n + (b_{n+1}a_{n+1} - c)x_n) \in \mathrm{Um}_n^r(R).$$

As $c \in I$ then $\overline{c} = \overline{0}$ and

$$(b_1 + b_{n+1}a_{n+1}x_1, \dots, b_n + b_{n+1}a_{n+1}x_n) \in \mathrm{Um}_n^r(\overline{R})$$

so st.r.$(R/I) \leq n =$ st.r.(R). □

Theorem 1.2. *For any ring R and any ideal I in $J(R)$:* st.r.$(R) =$ st.r.(R/I). *In particular,* st.r.$(R) =$ st.r.$(R/J(R))$.

Proof. In view of Proposition 1.2 we only need to show that st.r.$(R) \leq$ st.r.(R/I). Again let $\overline{R} = R/I$ and st.r.$(\overline{R}) = n$. Hence for any $b \in \mathrm{Um}_{n+1}^r(R)$ its image is again unimodular and by the stable range condition:

$$\overline{(b_1 + b_{n+1}u_1)x_1} + \dots + \overline{(b_n + b_{n+1}u_n)x_n} = \overline{1}$$

for some $\overline{x_1}, \dots, \overline{x_n}, \overline{u_1}, \dots, \overline{u_n}$. Then there is $c \in I$ such that

$$(b_1 + b_{n+1}u_1)x_1 + \dots + (b_n + b_{n+1}u_n)x_n = 1 + c \in U(R)$$

as I is in $J(R)$. Hence $(b_1 + b_{n+1}u_1, \dots, b_n + b_{n+1}u_n) \in \mathrm{Um}_n^r(R)$ and st.r.$(R) \leq$ st.r.(R/I). □

The following lemma will help to prove one important result concerning the stable range of matrix rings.

Lemma 1.1. *The rectangular matrix $A = \left(\begin{smallmatrix} 1 & u^T \\ 0 & A' \end{smallmatrix}\right)$ over a ring R is left invertible if and only if A' is so, where u^T is a row over R.*

Proof. Suppose that A is left invertible. Then there is a matrix B of appropriate size such that

$$I_n = BA = \begin{pmatrix} x & y^T \\ z & B' \end{pmatrix} \begin{pmatrix} 1 & u^T \\ 0 & A' \end{pmatrix} = \begin{pmatrix} x & xu^T + y^T A' \\ z & zu^T + B'A' \end{pmatrix}$$

for some $n \geq 1$. Hence $z = 0$ and $B'A' = I_{n-1}$.

Conversely, let $B'A' = I_{n-1}$. Then

$$\begin{pmatrix} 1 & -u^T \\ 0 & I_{n-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & B' \end{pmatrix} \begin{pmatrix} 1 & u^T \\ 0 & A' \end{pmatrix} = \begin{pmatrix} 1 & -u^T \\ 0 & I_{n-1} \end{pmatrix} \begin{pmatrix} 1 & u^T \\ 0 & I_{n-1} \end{pmatrix} = I_n$$

and A is left invertible as desired. □

Theorem 1.3. *For any ring R and any $k \geq 1$:*

$$\text{st.r.}(R_k) = 1 + \left\lceil \frac{\text{st.r.}(R) - 1}{k} \right\rceil,$$

where $\lceil x \rceil$ denotes least integer greater or equal x.

Proof. As any left unimodular row of length l over R_k can be considered as left invertible $lk \times k$ matrix then it is necessary and sufficient to prove that the $m-$stable range condition is equivalent to: for any left invertible $(m+k) \times k$ matrix B there exists $v \in R^k$ and left invertible $(m+k-1) \times k$ matrix B' such that

$$\begin{pmatrix} I_{m+k-1} & v \\ 0 & 1 \end{pmatrix} B = \begin{pmatrix} B' \\ u^T \end{pmatrix},$$

where u^T is the last row of B.

We use the induction on number k. For $k = 1$ the mentioned condition coincides with the stable range n condition. Suppose that for all n and l such that $k > l \geq 2$ we have proved the statement for $\text{st.r.}(r) = n$.

Remark 1.2. Suppose that A is an invertible matrix, i.e. $A \in GL_n(R)$ and a is its first column. Then $b^T a = 1$, where b^T is a first row of A^{-1}. So $a \in \text{Um}_n^l(R)$. However the converse isn't always true.

It is natural to ask: is every unimodular column (row) of length n a first column (row) of some invertible $n \times n$ matrix? Note that the answer is always affirmative for $n = 1$, and for $n = 2$

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Proposition 1.3. *Let R be a ring of stable range 1. If $aR = bR$ then $a = bu$, where u is an unit element of R.*

Proof. Since $aR = bR$ then $a = by$, $b = ax$ for some elements $x, y \in R$. Then $a = axy$ and $a(1-xy) = 0$.

Let $c = 1-xy$, then $xR + cR = R$ and $ac = 0$. Since st.r.$(R) = 1$ then $x + cv = u$ is an unit element of R for some element $v \in R$. Then $ax + acv = au$. Since $ac = 0$ then $b = ax = au$ and $bu^{-1} = a$. Proposition is proved. □

Proposition 1.4. *Let R be a ring. The following statements are equivalent:*

1. st.r.$(R) = 1$*;*

2. every left unit lifts modulo every left principal ideal;

3. every right unit lifts modulo every right principal ideal.

Proof. $(1) \Rightarrow (2)$ Let st.r.$(R) = 1$. Let $a, b, c \in R$ such that $ab - 1 \in Rc$, i.e. b is a left unit modulo the left principal ideal Rc. We are going to show that there exists a left unit $u \in R$ such that $b - u \in Rc$.

Let $x \in R$ be such that $ab - 1 = xc$. Then $ab - cx = 1$. Since st.r.$(R) = 1$ there exists $t \in R$ and unit w such that $b - txc = w$. Therefore $b - w \in Rc$, where w is a unit element and hence a left unit in R.

$(2) \Rightarrow (1)$ Let $a, b, c, t \in R$ be such that $Rb + Rc = R$ and $ab + tc = 1$. Then $ab - 1 \in Rc$. By hypothesis there exists a left unit $u \in R$ such that $b - u \in Rc$. Let $uv = 1$. Consider the left principal ideal $R(1 - vu)$. Since $(vu - 1) \in R(1 - vu)$ by our hypothesis there exists a left unit w such that $u - w \in R(1 - vu)$. Then there exists an element $s \in R$ such that $sw = 1$. Let $u - w = x(1 - vu)$. Multiplying on the right by v we have $(u - w)v = x(1 - vu)v$ which implies $1 - wv = 0$. Hence $wv = 1$ and v is a unit element of R and $vu = 1$.

Since $b - u \in Rc$ where u is a unit element we get that $b + \alpha c = u$ for some element $\alpha \in R$, i.e. st.r.$(R) = 1$.

$(1) \Rightarrow (3)$ The proof is similar. □

Theorem 1.4. *Let R be a ring in which every left principal ideal is a left annihilator for some element in R. Then the following statements are equivalent:*

1. st.r.$(R) = 1$*;*

2. if $aR = bR$ then there exist units $u, v \in R$ such that $au = b$ and $bv = a$.

Proof. (2) $\Rightarrow$ (1) By Proposition 1.4 it is suffices to show that every unit lifts modulo every left principal ideal in R.

Let x be a left unit that need to be lifted modulo the left principal ideal Ry, i.e. there exists $z \in R$ such that $zx - 1 \in Ry$. We would like to show that there exists a unit u such that $x - y \in Ry$. Due to the assumption about R both Ry and $R(xa)$ are left annihilators of some elements, that is. There exist $a, b \in R$ such that $Ry = l.\mathrm{Ann}(a)$ and $R(xa) = l.\mathrm{Ann}(b)$.

Since $zx - 1 \in Ry$ then $Rx + Ry = R$. For any $r \in R$ it is true that then

$$rx(ab) = (rxa)b = 0,$$

since $rxa \in Rxa = l.\mathrm{Ann}(b)$. Similarly

$$ry(ab) = ((ry)a)b = 0 \cdot b = 0$$

since $ry \in Ry = l.\mathrm{Ann}(a)$. Therefore $Rx \subseteq l.\mathrm{Ann}(ab)$ and $Ry \subseteq l.\mathrm{Ann}(ab)$.

Hence we have $R = Rx + Ry = l.\mathrm{Ann}(ab)$ and $ab = 0$, i.e. $a \in l.\mathrm{Ann}(b)$ and $Ra \in l.\mathrm{Ann}(b)$.

Also we have $l.\mathrm{Ann}(b) = Rxa \subseteq Ra$. Therefore $l.\mathrm{Ann}(b) = Rxa = Ra$. Then there exists a unit $v \in R$ such that $xa = va$ that implies $(x - v)a = 0$, so $x - v \in l.\mathrm{Ann}(a) = Ry$. By Proposition 1.4 we conclude $\mathrm{st.r.}(R) = 1$.

(1) $\Rightarrow$ (2) Follows from Proposition 1.3.

Theorem is proved. □

Proposition 1.5. *Let I be an ideal of a ring R. Then the following statements are equivalent:*

1. $\mathrm{st.r.}(R) = 1$*;*

2. $\mathrm{st.r.}(R/I) = 1$ *and* $\mathrm{st.r.}(R/r.\mathrm{Ann}(I)) = 1$*;*

3. $\mathrm{st.r.}(R/I) = 1$ *and* $\mathrm{st.r.}(R/l.\mathrm{Ann}(I)) = 1$.

Proof. (2) $\Rightarrow$ (1) By Second Isomorphism Theorem

$$\frac{R}{(I \cap r.\mathrm{Ann}(I)}\Big/\frac{I}{(I \cap r.\mathrm{Ann}(I)} \cong R/I,$$

$$\frac{R}{(I \cap r.\mathrm{Ann}(I)}\Big/\frac{I}{(I \cap r.\mathrm{Ann}(I)} \cong R/r.\mathrm{Ann}(I),$$

and we see that $\mathrm{st.r.}(R/(I \cap r.\mathrm{Ann}(I))) = 1$. As $(I \cap r.\mathrm{Ann}(I))^2 = 0$ we know that

$$I \cap r.\mathrm{Ann}(I) \subset J(R).$$

Therefore st.r.$(R) = 1$.

(1) $\Rightarrow$ (2) It is straight forward as always st.r.$(R) \geq$ st.r.(R/I).

(1) $\Leftrightarrow$ (3) can be proved in the same manner.

Theorem is proved. □

Proposition 1.6. *A ring R has stable range 1 if and only if*

$$aR + bR + cR = R$$

implies that $aR + (b + cr)R = R$ for some element $r \in R$

Proof. ($\Rightarrow$) Suppose $aR + bR + cR = R$. Then $ax + by + cz = 1$ for some elements $x, y, z \in R$. Hence $bR + (ax + cz)R = R$. Thus there is some $t \in R$ such that $b + (ax + cz)t = u$ where u is an unit element in R. Hence

$$axtu^{-1} + (b + czt)u^{-1} = 1.$$

Therefore $aR + (b + cr)R = R$, where $r = zt$.

($\Leftarrow$) It is clear as one can take $a = 0$.

Proposition is proved. □

Proposition 1.7. *A ring R is of stable range 2 if and only if $aR + bR + cR = R$ implies that $au + bv + cw = 1$ for some elements $u, v, w \in R$ such that $Ru + Rv = R$.*

Proof. ($\Rightarrow$) Since st.r.$(R) = 2$ then the equality $aR + bR + cR = R$ implies that $(a + cx)R + (b + cy)R = R$ for some elements $x, y \in R$. Hence

$$(a + cx)u + (b + cy)v = 1$$

for some elements $u, v \in R$ and after the simplification

$$au + bv + c(xu + yv) = 1,$$

that is $Ru + Rv = R$ as was desired.

($\Leftarrow$) Let $aR + bR + cR = R$ implies that $au + bv + cw = 1$ for some elements $u, v, w \in R$ such that $Ru + Rv = R$. Then $xu + yu = w$ for some elements $x, y \in R$. Since $au + bv + cw = 1$ then after the substitution we obtain

$$au + bv + cxu + cyv = (a + cx)u + (b + cy)v = 1,$$

i.e. $(a + cx)R + (b + cy)R = R$ and st.r.$(R) = 2$. Proposition is proved. □

1.2 Ring theoretical constructions and their stable range

1.3 Completion to invertible matrix

1.4 Stabilization in K-theory

1.5 Examples and stable range calculations