Chapter 2
Rings ruled by units, annihilators and idempotents

2.1 Von Neumann regular rings

2.2 Exchange, clean and idempotent stable range one rings

Definition 2.1. An element a in a ring R is called *clean* if it can be written as the sum of a unit and an idempotent of the ring R. A ring R is said to be *clean* if every its element is clean.

Studying the problem of idempotents lifting and exchange rings Nicholson [47] has introduced the notion of clean ring. In this section we are going to review some their useful properties.

Next two results were proved by Anderson and Camillo [1].

Proposition 2.1. *Every homomorphic image of a clean ring is clean.*

Proof. Since the multiplication operation is preserved by any ring homomorphism, the homomorphic image of a unit and idempotent is again a unit and an idempotent respectively.

The addition operation is also preserved by any every ring homomorphism, so the image of clean element is clean, as was desired. □

Proposition 2.2. *A direct product of rings* $\prod R_i$ *is clean if and only if each ring* R_i *is clean.*

Proof. Since the multiplication operation in a direct product of rings is defined componentwise, an element in a direct product of rings is a unit (idempotent) if and only if each of its components is an unit (idempotent). Since the addition operation in a direct product of rings is also defined componentwise the simple computations implies the desired property. □

The next result is due to Han and Nicholson [31].

Theorem 2.1. *A full matrix ring $M_n(R)$ is clean if the underlying ring R is clean.*

Proof. Han and Nicholson [31] have shown that if the identity 1 of a ring R can be written as a finite sum

$$1 = e_1 + e_2 + \ldots + e_n$$

of mutually orthogonal idempotents e_i such that each corner ring $e_i R e_i$ is clean then R is a clean ring itself.

Since the set of matrix units E_{ii}, for $i \in \{1, 2, \ldots, n\}$ is a complete set of orthogonal idempotents of $M_n(R)$ and each corner ring $E_{ii} M_n(R) E_{ii}$ is isomorphic to R then by the mentioned above property $M_n(R)$ is a clean ring. □

Definition 2.2. For an ideal I of a ring R we say that *idempotents lift modulo I* if for each element $x \in R$ such that $x - x^2 \in I$ there is some idempotent e of R with property $e - x \in I$. A ring R is an *exchange ring* if idempotents lift modulo every left (equivalently right) ideal I of R.

Proposition 2.3. *Every clean ring is an exchange ring.*

Proof. Let R be a clean ring and I is its arbitrary left ideal. Suppose that $x - x^2 \in I$ for some element $x \in R$. It suffices to show that there is some idempotent $e \in R$ such that $e - x \in I$. Since R is a clean ring then $x = u + f$ for some unit u and idempotent f of R. Then

$$e = u^{-1}(1 - f)u$$

is an idempotent of R such that

$$e - x = u^{-1}(x - x^2) \in I$$

as was desired. Therefore every clean ring is an exchange one. □

Proposition 2.4. *Let R be a ring and x its element. The following statements are equivalent:*

1. *there exists an idempotent $e \in R$ such that $e - x \in R(x - x^2)$;*

2. *there exists an idempotent $e \in Rx$ and $c \in R$ such that $(1 - e) - c(1 - x) \in J(R)$;*

3. *there exists an idempotent $e \in Rx$ such that $R = Re + R(1 - x)$;*

4. *there exists an idempotent $e \in Rx$ such that $1 - e \in R(1 - x)$.*

Proof. (1) $\Rightarrow$ (2) If $e - x = r(x - x^2)$ then $1 - e = (1 - rx)(1 - x)$.
(2) $\Rightarrow$ (3) Given (2) it is clear that $e + c(1 - x)$ is a unit.
(3) $\Rightarrow$ (4) Given (3) write $1 = te + s(1 - x)$ and define $f = e + (1 - e)te$. Then $f^2 = f \in Rx$ and $1 - f = (1 - e)s(1 - x) \in R(1 - x)$.
(4) $\Rightarrow$ (1) Given (4), we can write $e - x = e(1 - x) + (1 - e)x \in R(x - x^2)$. □

The next result is an immediate consequence of Proposition 2.4.

Proposition 2.5. *Every homomorphic image of an exchange ring is an exchange ring.*

Proposition 2.6. *A ring R is an exchange ring if and only if $R/J(R)$ is an exchange ring and idempotents can be lifted modulo $J(R)$.*

Proof. Assume that $R/J(R)$ is an exchange ring and let $x \in R$. Denote by $\overline{x} = x + J(R)$ and $\overline{R} = R/J(R)$. There exists $\overline{a}^2 = \overline{a} \in \overline{R}\overline{x}$ and $\overline{c} \in \overline{R}$ such that

$$\overline{1} - \overline{a} = \overline{c}(\overline{1} - \overline{x}).$$

We may assume that $a \in Rx$. Choose idempotent f such that $\overline{f} = \overline{a}$. Then $u = 1 - f + a$ is a unit in R and so

$$e = u + fu = u^{-1}fa$$

satisfies $e \in Rx$ and $e^2 = e$. Since $\overline{e} = \overline{f} = \overline{a}$ it follows that

$$(1 - e) = c(1 - x) \in J(R)$$

so R is an exchange ring by Proposition 2.4.
The converse statement follows from Definition 2.2 and Proposition 2.5. □

Notice that Proposition 2.5 remains true if $J(R)$ is replaced by any ideal $I \subseteq J(R)$.

Definition 2.3. A ring R is *semiregular* if $R/J(R)$ is a von Neumann regular ring and idempotents can be lifted modulo $J(R)$.

Proposition 2.7. *Every semiregular ring is an exchange ring.*

Proof. We may assume that R is a von Neumann regular ring by Proposition 2.5. If $x \in R$ choose $y \in R$ such that $xyx = x$ and write $f = yx$. If we denote

$$e = f + (1 - f)xf$$

then $e^2 = e$ and $e \in Rx$ and

$$1 - e = (1 - f)(1 - x).$$

The desired result follows from Proposition 2.4. □

Proposition 2.8. *A ring with central idempotents is clean if and only if it is an exchange ring.*

Proof. If $x = e + u$, where e is an idempotent and u is a unit then

$$u(x - u^{-1}(1 - e)u) = ue - u^2 - u + eu = x^2 - x$$

and result follows from statement (1) of Proposition 2.4.

If R is an exchange ring and $x \in R$ choose idempotent $e \in Rx$ with $(1 - e) \in R(1 - e)$. If $e = ax$ we may assume $ea = a$ so that $axa = a$. If idempotents are central then

$$xa = x(ax)a = xa(ax) = ax.$$

Similarly write $1 - e = b(1 - x)$ where $(1 - e)b = b$ and $b(1 - x) = (1 - x)b$. Easy calculations show that $a - b$ is the inverse of $x - (1 - e)$. □

Definition 2.4. A ring R is said to be *reduced* if it has no (nonzero) nilpotent elements.

Proposition 2.9. *A reduced ring is a ring with central idempotents.*

Proof. Let R be a reduced ring and e be an idempotent in R. For every $x \in R$ we have

$$(ex(1 - e))^2 = 0, \ ((1 - e)xe)^2 = 0.$$

Since R is reduced then $ex(1 - e) = 0$ and $(1 - e)xe = 0$ so $ex = exe = xe$. □

Definition 2.5. A ring R is called a *potent* ring if idempotents can be lifted modulo $J(R)$ and every left (right) ideal not contained in $J(R)$ contains a nonzero idempotent. A ring R is said to be *semipotent* if each left (right) ideal of R that is not contained in its Jacobson radical contains a nonzero idempotent.

It is well known that this notion is left-right symmetric. Obviously, any potent ring is semipotent.

Proposition 2.10. *Every exchange ring is potent.*

Proof. It suffices to show that there is a nonzero idempotent in Rx for each $x \neq J(R)$. Suppose that $x \in R$. Then there exists an idempotent $e \in R$ such that $e \in Rx$. Then we have that $e = 0$. Give $a \in R$ choose idempotent $\phi \in Rax$ such that $(1-e) \in R(1-ax)$. Then $\phi = 0$ and so $1 \in R(1-ax)$. This means $x \in J(R)$. □

We define a commutative ring

$$R(\mathbb{Q},\mathbb{Z}) = \{q_1,\ldots,q_n,z_1,\ldots,z_n \mid n \geq 1, q_i \in \mathbb{Q}, z_i \in \mathbb{Z}\}.$$

Obviously, $R(\mathbb{Q},\mathbb{Z})$ is a ring (with componentwise operations). Every nonzero ideal of a ring R contains a nonzero idempotent, so $R(\mathbb{Q},\mathbb{Z})$ is a commutative potent ring with $J(R) = 0$ that is not an exchange ring.

Proposition 2.11. *If R is an exchange ring and e is an idempotent of R. Then the corner ring eRe is an exchange ring.*

Proof. If $x \in eRe$ choose idempotent f such that $f \in Rx$ and $1-f \in R(1-x)$. Then $fe = f$ so

$$(ef)^2 = ef \in eRex$$

and

$$e - ef = e(1-f)e \in eRe(e-x).$$

□

Proposition 2.12. *Let R be an exchange ring and suppose that*

$$x_1 + x_2 + \ldots + x_n = 1$$

in R. Then there exist orthogonal idempotents $e_1,\ldots,e_n$ such that $e_i \in Rx_i$ for each i and

$$e_1 + e_2 + \ldots + e_n = 1.$$

Proof. Assume $n \geq 2$ and proceed by the induction. Given

$$x_1 + x_2 + \ldots + x_n + x_{n+1} = 1$$

choose an idempotent $f \in R(x_1 + x_2 + \ldots + x_n)$ with $1 - f \in Rx_{n+1}$. Write

$$f = r(x_1 + x_2 + \ldots + x_n).$$

Since fRf is exchange, choose by induction orthogonal idempotents $f_1, f_2, \ldots, f_n$ in fRf such that $f_i \in (fRf)(rx_i f)$ for each i and $f_1 + f_2 + \ldots + f_n = f$.

For each i we can write $f_i = fr_ifrx_if$ and define $e_i = f_ir_ifrx_i$. Then

$$e_ie_j = (f_ir_ifrx_i)(f_jr_jfrx_j) = f_if_j(r_jfrx_j)$$

holds for each i and j so these e_i are orthogonal idempotents. Write $e = -(e_1 + e_2 + \ldots + e_n) + 1$. Since $e_i \in Rx_i$ where $1 \leq i \leq n$ it remains to prove $1 - e \in Rx_{n+1}$. But $e_if = f_i^2$ for each i and so $ef = f$. Consequently

$$1 - e = (1 - e)(1 - f) \in Rx_{n+1}$$

(see [47]). □

Definition 2.6. An element x in a ring R is said to be *quasiregular* if there is some element $y \in R$ such that

$$x + y = xy = yx.$$

Proposition 2.13. *An element x in a ring R is quasiregular if and only if $1 - x$ is a unit on R.*

Proof. By the assumption we have $(1 - y)(1 - x) = (1 - x)(1 - y) = 1$. □

Proposition 2.14. *Every clean ring is semipotent.*

Proof. Let R be any clean ring and let I be any left ideal of R such that $I \nsubseteq J(R)$. We need to show that I contains a nonzero idempotent.

It is well known that every left (right) ideal of ring R that is not contained in its Jacobson radical must contain an element that is not quasiregular. Since $I \nsubseteq J(R)$, there exist some $a \in I$ that is not quasiregular.

We have $a = u + e$ for some unit u and idempotent e of R. Since a is not quasiregular, it follows that $e \neq 1$. Then $u^{-1}(1 - e)u$ is idempotent. Moreover, notice that

$$u^{-1}(1 - e)u = u^{-1}(1 - e)(a - e) = u^{-1}(1 - e)a \in I.$$

It follows $u^{-1}(1 - e)u$ is nonzero idempotent of I. This completes the proof. □

By Proposition 2.10 and Proposition 2.3 we have the following result.

Proposition 2.15. *Every clean ring is potent.*

Gathering all results we obtain the following chain of ringś classes inclusions:

$$\text{clean} \ \Rightarrow \ \text{exchange} \ \Rightarrow \ \text{potent} \ \Rightarrow \ \text{semipotent}$$

This implications are known to be irreversibe.

In particular, Camillo and Yu [7] have proved that the ring in well-known Bergman's [30] example is an exchange ring that is not clean. Meanwhile, Nicholson [47] has shown that a potent ring need not to be an exchange one and Nicholson and Zhou [51] have constructed and example of semipotent ring that is not a potent one.

Definition 2.7. A ring R is said to have an *idempotent stable range 1* if for any elements $a,b \in R$ such that $Ra+Rb=R$ there exists some idempotent $e \in R$ such that $a+eb$ is a unit of R.

Proposition 2.16. *Every ring that has an idempotent stable range 1 is clean.*

Proof. Let R be a ring of idempotent stable range 1 and $a \in R$. Obviously $aR + (-1)R = R$. Then exists some idempotent e of R such that an element $a+(-1)e = u$ is a unit of R. Then $a = u+e$. □

Proposition 2.17. *Let R be a ring whose idempotents are central. Then the following are equivalent:*

1. R has an idempotent stable range 1;

2. R is a clean ring;

3. R is an exchange ring.

Proof. The implications $(1) \Rightarrow (2) \Rightarrow (3)$ are straight forward due to Proposition 2.16 and Proposition 2.8.

$(3) \Rightarrow (1)$ Assume that R is not a ring of idempotent stable range 1. Then there exist $a,b \in R$ with $Ra+Rb=R$, while $a+eb$ is not a unit of R for any idempotent $e \in R$.

Let A be a two-sided ideal of a ring R such that $a+eb$ is not a unit modulo A for any idempotent $e \in R$. Denote by S the set of all such ideals A of R. It is obviously, that S is a nonempty inductive set since $(0) \in S$.

Using Zorn's Lemma we can prove that there exists some two-sided ideal Q of R such that it is maximal in S. By the maximality of Q we will show that R/Q is indecomposable as a ring. Take any $x \in R/Q$. Since R is an exchange ring, so is R/Q. By Proposition 2.4 there is an idempotent $e \in R/Q$ such that

$$e \in xR/Q, \quad (1-e) \in (1-x)R/Q$$

and an idempotent $f \in R/Q$ such that

$$f \in R/Qx, \quad (1-f) \in R/Q(1-x).$$

Since idempotents in R/Q can be lifted modulo Q we may assume that e and f are both central idempotents in R/Q. So $e = 0$ or $e = 1$ and $f = 0$ or $f = 1$. Thus we see that x or $1-x$ is right invertible in R/Q. Similarly x or $1-x$ is left invertible.

Assume that $x \in R/Q$ is not invertible. If x is not left invertible in R/Q, then rx is not left invertible for any $r \in R/Q$. Thus $1-rx$ is left invertible. This shows that $x \in J(R/Q)$. If x is not right invertible in R/Q, similarly to the discussion above, we have $x \in J(R/Q)$. This implies that R/Q is local.

A local ring is obviously a ring of idempotent stable range 1. The R/Q has idempotent stable range 1, a contradiction. Then R has idempotent stable range 1. □

Lemma 2.1. *The units, idempotents and quasiregular elements of any ring are clean.*

Proof. Any unit u can be written as the sum of a unit and idempotent: $u = u + 0$. Moreover any $e^2 = e$ with complement $f = 1-e$ can be written as the sum of a unit and an idempotent: $e = (f-e) + f$, since the difference $e-f$ is a square root of 1 and hence a unit.

Finally any quasiregular element x can be written as the sum of a unit and an idempotent: $x = (x-1) + 1$ since an element x is quasiregular if and only if $x-1$ is a unit. □

Corollary 2.1. *Every division ring, boolean ring and local ring is clean.*

Proof. All division rings, boolean rings and local rings consist entirely of units, idempotents and quasiregular elements that are clean by Lemma 2.1. □

As an immediate corollary, we have the following result.

Proposition 2.18. *A domain is clean if and only if it is a local.*

Therefore every semilocal domain that is not local is an example of stable range 1 ring which is not idempotent stable range 1 ring.

Definition 2.8. A left R-module M is said to have the *exchange property* (see Crawly and Jònsson [17] and later Warfield [70]) if for any module X and decompositions

$$X = M' \oplus Y = \bigoplus_{i \in I} N_i$$

where $M' \cong M$, there exist submodules $N_i' \subseteq N_i$ for each i such that

$$X = M' \oplus (\bigoplus_{i \in I} N_i').$$

If this condition holds for any finite set I (equivalently for $|I| = 2$) the module M is said to have the finite exchange property.

Theorem 2.2. *[47] Let R be a ring. The following conditions are equivalent for a left R-module M:*

1. $\mathrm{End}_R(M)$ *is an exchange ring;*

2. M has the finite exchange property.

Definition 2.9. A commutative ring R is called a *Gelfand ring* if whenever $a+b = 1$ there exist elements $r, s \in R$ such that

$$(1+ar)(1+bs) = 0.$$

Definition 2.10. A ring is called a *PM-ring* if every prime ideal is contained in a unique maximal ideal.

It is known that for any topological space X the ring $C(X)$ consisting of all real-valued continuous functions on X under the pointwise operations is always a *PM*-ring ????. Other obvious examples of *PM*-rings are commutative von Neumann regular rings, local rings and zero-dimensional rings.

Theorem 2.3. *A commutative ring R is a PM-ring if and only if R is a Gelfand ring.*

Proof. First, let us assume that R is a *PM*-ring and let $a, b \in R$ be such that $a+b = 1$. Consider the multiplicatively closed sets

$$S_1 = \{1+ar | r \in R\},\ S_2 = \{1+bs | s \in R\}.$$

We claim that the multiplicatively closed set $S = S_1 S_2$ contains 0. If this is not true, there would be a prime ideal $P \in \mathrm{Spec}(R)$ with $P \cap S = \emptyset$.

Moreover, the ideal $P + Ra$ is not the whole ring because if $1 = p + ar$, where $p \in R$, $r \in R$ would imply

$$p = 1 - ar \in S_1 \subset S,$$

that is impossible since $P \cap S = \emptyset$.

Therefore, there exists a maximal ideal $M \in \mathrm{mspec}(R)$ that contains $P+Ra$. Similarly, there exists a maximal ideal $N \in \mathrm{mspec}(R)$ containing $P+bR$. Notice, that $M \neq N$, otherwise $1 = a+b \in M = N$, that is impossible. Since

$$P \subset P+aR \subset M, \; P \subset P+bR \subset N,$$

we have that P is belongs to two different maximal ideals which contradicts to the assumption.

So S contains 0.

Conversely, assume that R is a Gelfand ring and R is not a PM-ring. Let P be a prime ideal that is contained in two distinct maximal ideals $M, N \in \mathrm{mspec}(R)$. Since $M+N = R$ there exist $a \in M$ and $b \in N$ such that $a+b = 1$. Then there exist elements $r, s \in R$ such that $(1+ar)(1+bs) = 0$. Since P is a prime ideal and $0 \in P$ it follows that $1+ar \in P$ or $1+bs \in P$ and then $M = R$ or $N = R$. The obtained contradiction proves the statement. □

As the corollaries of Theorem 2.3, we have the following:

1) a commutative von Neumann regular ring R is a Gelfand ring since the equality $a+b=1$ implies that $(1-ax)a = (1-ax)(1-b) = 0$, where $axa = a$ for some $x \in R$.

2) Let R be a zero-dimensional ring. If N is the nilradical of R, then R/N is obviously a von Neumann regular ring and therefore the equality

$$\overline{a}(\overline{1} - \overline{ax}) = \overline{0}$$

lifts to $a(1-ax) \in N$ in R or $a^n(1-ax)^n = 0$ for some $n \in \mathbb{N}$.

3) In a local ring R where $a+b=1$ implies that at least a or b is unit and thus we have the equality

$$(1+ar)(1+bs) = 0$$

with $r = a^{-1}$ or $s = b^{-1}$, so local ring is a Gelfand one.

Proposition 2.19. *A commutative clean ring is a PM-ring.*

Proof. Let R be a commutative clean ring and $P \in \mathrm{Spec}R$. Then R/P is a clean domain. By Proposition 2.18, R/P is local domain, then R/P is a PM-ring. By the corresponence theorem a ring R is a PM-ring. □

2.3 Potent and semipotent rings

2.4 VNL, NJ and semiregular rings

2.5 PM and Gelfand rings

2.6 Semihereditary and morphic rings

It is a well-known theorem of Erlich [20] that a map ϕ in an endomorphism ring of R-module M is unit-regular if and only if it is regular and

$$M/\mathrm{Im}(\phi) \cong \mathrm{Ker}(\phi).$$

We focus on the case $M =_R R$, so if $\alpha = \cdot a :_R R \to_R R$ is a right multiplication by the element $a \in R$, the condition above becomes

$$R/Ra \cong l(a).$$

where denotes the left annihilator of an element a.

Definition 2.11. An element a of a ring R is called *a left (right) morphic* if $R/Ra \cong l(a)$ $(R/aR \cong r(a))$. The ring R is called *a left (right) morphic ring* if every its element is left (right) morphic.

Lemma 2.2. *For a ring R and element $a \in R$ conditions are equivalent:*

1. a is a left morphic element;

2. there exists $b \in R$ such that $Ra = l(b)$ and $l(a) = Rb$;

3. there exists $b \in R$ such that $Ra = l(b)$ and $l(a) \cong Rb$.

Proof. Given (1) let $\phi : R/Ra \to l(a)$ be an isomorphism and put $b = \phi(\overline{1})$, where $\overline{1} = 1 + Ra$. Then $Rb = \mathrm{Im}(\phi) = l(a)$, because ϕ is onto, and $l(b) = Ra$ because ϕ is one-to-one. This proves (1)⇒(2) and (2)⇒(3) is clear. But if (3) holds, then $R/Ra = R/l(b) \cong Rb \cong l(a)$. □

Lemma 2.3. *If a is a left morphic element in a ring R the same is true about au and ua for every unit u in R.*

Proof. Choose $b \in R$ such that $Ra = l(b)$ and $Rb = l(a)$. Then

$$R(ua) = Ra = l(b) = l(bu^{-1}), \quad R(bu^{-1}) = l(a)u^{-1} = l(ua),$$

so ua is left morphic. Again,

$$R(au) = l(b)u = l(u^{-1}b), \quad R(u^{-1}b) = R(b) = l(a) = l(au)$$

and ua is again left morphic. □

Definition 2.12. An element $a \in R$ is called a *von Neumann (unit-) regular* if $axa = a$ for some (unit) $x \in R$. The ring R is called a *von Neumann (unit-) regular ring* if every its element has this property.

Remark 2.1. If a is unit-regular element then, say $aua = a$, where u is a unit. If we take $e = ua$ then $a = u^{-1}e$ is left morphic, because $e^2 = e$ and every idempotent is a left morphic element.

Proposition 2.20. *Every unit-regular ring is left and right morphic.*

Example 2.1. The converse to the latter statement is false: $\mathbb{Z}_4$ is left morphic but it is not unit-regular ring.

Proposition 2.21. *If $a \in R$ is left morphic the following statement are equivalent:*

1. $l(a) = 0$*;*

2. $Ra = R$*;*

3. $a \in U(R)$.

Proof. Choose $b \in R$ such that $Ra = l(b)$ and $l(a) = Rb$. Then $l(a) = 0$ if and only if $b = 0$ and $Ra = R$ if and only if $b = 0$. This proves that (1)⇔(2) and this certainly are equivalent to (3). □

Example 2.2. Thus a polynomial ring $R[x]$ is never left or right morphic because $l(x) = 0$ and $x \notin U(R[x])$, and the only left (right) morphic domains are the division rings.

The following result gives another source of examples of left morphic rings.

Proposition 2.22. *If a ring R has a unique left ideal $I \neq 0$ then R is left morphic.*

Proof. Clearly $I = J(R)$ is the Jacobson radical of R. If $a \in R$ then we must show that a is left morphic. Obviously, R is a local ring and $I = Ra$ by the hypothesis. Furthermore, $l(a) \neq R$ because $a \neq 0$ and we are done if we can show that $l(a) \neq 0$ (then $l(a) = I$ and so $l(a) = Ra$).

But if $l(a)$, then $r \mapsto ra$ is an isomorphism $R \cong Ra$, and it follows that $0 \subset Ia \subset I$, contrary to the hypothesis. □

In the following we will consider only the commutative rings and left annihilator $l(a)$ we denote by $\mathrm{Ann}(a)$.

Definition 2.13. A commutative left morphic ring we will call simply *a morphic ring*.

Theorem 2.4. *Let R be a commutative morphic ring. Then:*

1. finite intersections of principal ideals of R are again principal;

2. R is a Bezout ring.

Proof. Let $b \in R$ is such that $Rb = \mathrm{Ann}(x)$where there is $x \in R$. Then let $\mathrm{Ann}(ax) = Ry$ for some $y \in R$. We will show that $Ra \cap Rb = Rya$. To see that $Ra \cap Rb \subseteq Rya$ let take $w \in Ra \cap Rb$. Suppose $w = sa = tb$ for some $s,t \in R$.

It suffices to show that $s \in Ry$, that is $s \in \mathrm{Ann}(ax)$. But $s(ax) = (tb)x = 0$, because $b \in \mathrm{Ann}(x)$. Finally, we show that $Rya \subseteq Ra \cap Rb$. Clearly $Rya \subseteq Ra$, so it remains only to show that $ya \in Rb$, that is $ya \in \mathrm{Ann}(x)$. But $(ya)x = 0$, because $y \in \mathrm{Ann}(ax)$.

(2) Given $a,b \in R$ let $Ra = \mathrm{Ann}(x)$ for some $x \in R$ and let $Rbx = \mathrm{Ann}(y)$ for some $y \in R$. Since R is a morphic ring it is enough to show that $Ra + Rb = \mathrm{Ann}(xy)$. In fact $a(xy) = (ax)y = 0 \cdot y = 0$ and $b(xy) = (bx)y = 0$, so $Ra + Rb \subseteq \mathrm{Ann}(xy)$. For the converse inclusion let $r \in \mathrm{Ann}(xy)$. Then $rx \in \mathrm{Ann}(y) = Rbx$, say $rx = sbx$, where $s \in R$. Hence $r - sb \in l(x) = Ra$, so $r \in Ra + Rb$. This proves the fact that $\mathrm{Ann}(xy) \subseteq Ra + Rb$, as required. □