Chapter 4
Adequacy

4.1 Adequate and coadequate elements

All rings in this section will be usually the rings in which any nonzero prime ideal is contained in a unique maximal ideal.

Definition 4.1. An element a of commutative ring R is called *adequate to an element b* and b is called *coadequate to an element* a(${}_a\mathrm{A}_b$ in notation) if one can find elements $r, s \in R$ such that

1. $a = rs$;

2. $rR + bR = R$;

3. for any nonunit divisor s' of s: $s'R + bR \neq R$.

For a fixed element $b \in R$ we introduce a set ${}_*\mathrm{A}_b = \{a \in R \mid {}_a\mathrm{A}_b\}$. Similarly fixing $a \in R$ we consider a set ${}_a\mathrm{A}_* = \{b \in R \mid {}_a\mathrm{A}_b\}$. In the following we will describe the properties of such sets.

Proposition 4.1. *For any commutative ring R the set ${}_*A_b$ is multiplicatively closed.*

Proof. Since $1 \in {}_*\mathrm{A}_b$ then ${}_*\mathrm{A}_b \neq \varnothing$. So we only need to prove that ${}_*\mathrm{A}_b$ is closed under taking products of its elements. Let $a, d \in {}_*\mathrm{A}_b$. Then there are elements $r, m, t, l \in R$ such that $a = rm$, $d = tl$ where

$$rR + bR = R, \ \ tR + bR = R$$

and for any elements m', l' such that $mR \subset m'R \neq R$ and $lR \subset l'R \neq R$ implies $m'R + bR \neq R$, and $l'R + bR \neq R$.

Then $rtR + bR = R$ and for any $r' \in R$ such that $mlR \subset n'R \neq R$ we obtain

$$n'R + bR \subseteq (n'R + mR + bR) \cap (n'R + lR + bR) \neq R.$$

Thus $n'R + bR \neq R$. So $ad \in {}_*\mathrm{A}_b$ as was desired. □

Proposition 4.2. *If R is a commutative Bezout domain then ${}_*A_b$ is a saturated set.*

Proof. Let $a \in {}_*\mathrm{A}_b$ and $a = dx$ for some elements $x \in R$, $d \in R$. By the definition of the elements $a \in {}_*\mathrm{A}_b$ there are elements $r, s \in R$ such that $a = rs$ where $rR + bR = R$ and $sR \subset s'R$ implies $s'R + bR \neq R$. Let $dR + rR = hR$. Since R is a Bezout domain then there are some elements $d_0, r_0 \in R$ such that

$$d = hd_0,\ r = hr_0,\ d_0R + r_0R = R.$$

This implies the equality $d_0u + r_0v = 1$ for some elements $u, v \in R$. Then $a = hd_0x = hr_0s$ implies the equalities $d_0x = r_0s$ and $sd_0u + sr_0v = s$, since R is a ring without zero divisors. Thus we obtain the equality $d_0(su + xv) = s$ i.e. the inclusion $sR \subset d_0R$ holds.

If $d_0R \subseteq d_0'R \neq R$ then d_0' can be used as s' and we have

$$d_0'R + bR \neq R.$$

Inasmuch

$$R = rR + bR \subset hR + bR$$

we see that the decomposition $d = d_0h$ satisfies all conditions of the adequate element's definition $d \in {}_*\mathrm{A}_b$. So any divisor of element from ${}_*\mathrm{A}_b$ is again in ${}_*\mathrm{A}_b$ as was desired. □

Proposition 4.3. *If R is a commutative Bezout domain then ${}_aA_*$ is a multiplicatively closed set.*

Proof. Clearly ${}_a\mathrm{A}_1$ and thus $1 \in {}_a\mathrm{A}_*$. So, ${}_a\mathrm{A}_*$ is nonempty. Next, suppose that $x, y \in {}_a\mathrm{A}_*$. Then there are $r_1, r_2, s_1, s_2 \in R$ such that

$$a = r_1s_1 = r_2s_2,\ r_1R + xR = r_2R + yR = R,\ s_1'R + xR \neq R,\ s_2R + yR \neq R$$

for any non invertible divisors s_1', s_2' of s_1, s_2 respectively.

Let $rR = r_1R + r_2R$ and $r_1 = ru$, $r_2 = rv$ for some $r, u, v \in R$. Note that $uR + vR = R$ as R is a domain. Furthermore, $rR + xR = r_1R + r_2R + xR = R$ and $rR + yR = r_1R + r_2R + yR = R$.

Suppose that $rR + s_1R = hR \neq R$. Then $hR + xR \supseteq rR + xR = R$. But since h is a non invertible divisor of s_1 then $hR + xR \neq R$. The obtained contradiction proves the fact that $rR + s_1R = R$.

Moreover, the equality $rus_1 = rvs_2$ implies $us_1 = vs_2$ again since R is a domain. Hence $s_2R \subseteq uR$. If we take $s = us_1$ then $a = rs$ and

$$R = (xR + rR)(yR + rR) = abR + rR,$$

$$s'R + xyR \subseteq (s'R + xR) \cap (s'R + yR) \subseteq (s'R + s_1R + xR) \cap (s'R + s_2R + yR) \neq R,$$

where s' is arbitrary non invertible element such that $s'R \supseteq us_1R \supset s_1s_2R$. Thus ${}_a\mathrm{A}_{xy}$ and ${}_a\mathrm{A}_*$ is multiplicatively closed. □

Definition 4.2. An element $a \in R$ is called *adequate* if for every element $b \in R$: ${}_a\mathrm{A}_b$. If b is a nonzero element then it is called *coadequate* if for any element $a \in R$: ${}_a\mathrm{A}_b$ A commutative ring is called an *adequate ring* if all its nonzero elements are adequate. If even a zero element of adequate ring is an adequate element such ring is called *everywhere adequate*.

Although any everywhere adequate ring is an adequate ring the converse is not always true. For example, a ring of integers $\mathbb{Z}$ is adequate but not everywhere adequate ring.

Example 4.1. In any commutative ring R the idempotents are coadequate.
In fact if $e^2 = e \in R$ then for any $a \in R$ there are the decompositions

$$a = (1 - e + ea)(e + a - ea),\ e = e(e + a - ea).$$

Since $(1 - e + ea) \cdot +e(1 - a) = 1$ then $(1 - e - ea)R + eR = R$. Moreover if s' is a non invertible divisor of $e + a - ea$ then $s'R + eR \subseteq (e + a - ea)R \neq R$. Hence ${}_a\mathrm{A}_e$ as was desired.

Since any commutative principal ideal domain is a factorial domain ???? we have the following result.

Proposition 4.4.

Theorem 4.1. *A commutative principal ideal domain is an adequate domain.*

Proof. Let R be a commutative principal ideal domain and a be a nonzero element of R. Consider any element $c \in R$. If $c = 0$ then obviously ${}_c\mathrm{A}_a$. Since every commutative principal ideal domain is a factorial domain then for in a case of nonzero c there are decompositions

$$a = u_1 {p_1}^{k_1} \ldots {p_n}^{k_n},\ \ c = u_2 {p_1}^{s_1} \ldots {p_n}^{s_n}$$

where u_1 and u_2 are units of R, $p_1, \ldots, p_n$ – prime elements of R and $k_1, \ldots, k_n, s_1, \ldots, s_n \in \mathbb{N} \cup \{0\}$.

Let $p_1, \ldots, p_t$ are the prime elements which are divisors of element a and

$$p_iR + cR = R,$$

where $i = 1, 2, \ldots, t$. Let $r = p_1{}^{k_1} \ldots p_t{}^{k_t}$ then $s = \frac{a}{r}$ is such element that every prime divisor of s is a divisor of c. Therefore $a = rs$, where $rR + cR = R$ and $s'R + cR \neq R$ for each nonunit divisor s' of s. Theorem is proved. □

Example 4.2. Let R be a ring of entire functions on the complex plane and $f, g \in R$. Let (C_i, n_i) be the set of common zeroes with their multiplicities. By Mittag-Lefler theorem there is an entire function h having precisely these zeroes. Then function $a = \frac{f}{h}$ that divides f is coprime with g, and is such that for any nonunit b dividing $\frac{f}{a}$ has no common zero with g.

Also the function

$$K = \left(\frac{f}{h}\right)^2 + \left(\frac{g}{h}\right)^2$$

is always nonzero, so a is unit in R, thus the function

$$h = \left(\frac{f}{hK}\right) f + \left(\frac{g}{hK}\right) g$$

is the greatest common divisor of f and g:

$$fR + gR = hR.$$

This proves that R is a commutative Bezout domain. Taking $r = \frac{f}{h}$ and $s = h$, we have $f = rs$, where $rR + gR = R$ and $s'R + gR \neq R$ for each nonunit divisor s' of s, hence R is an adequate domain.

Therefore we have proved that a ring of entire functions on the complex plane is a commutative adequate Bezout domain. Obviously, R is not a principal ideal domain.

Theorem 4.2. *Every a commutative von Neumann regular ring is an everywhere adequate ring.*

Proof. Let R be a commutative von Neumann regular ring. First we show that R is a Bezout ring. If $a^2x = a$ then $e = ax$ is an idempotent and $aR = eR$. Now, let's prove that every finitely generated ideal is principal. For $b^2y = b$ we have the equalities

$$f = by, \quad f^2 = f.$$

If we denote $d = e + f - ef$ then $a = ad$, $b = bd$ and

$$d \in eR + fR = aR + fR = aR + bR \subseteq dR + dR = dR.$$

Therefore $dR = aR + bR$, and every element is a unit multiple of an idempotent.

Now we will show that x can be chosen invertible in $a^2x = a$.

Let x satisfies $a^2x = a$ and z satisfies $x^2z = x$. Since $axz = (a^2x)xz = a^2x = a$, then denoting $u = 1 + x - xz$ we have $a^2u = a$.

Obviously, $uR + xR = R$. But $xu = x^2$, whence x belongs to every maximal ideal that contains u. It follows that u is a unit.

To prove that R is an adequate ring let $a, b \in R$. By the von Neumann regularity of R there are idempotents $e, f \in R$ such that $e = au$, $f = bv$, where u, v are some invertible elements of R.

As was noted above, $dR = eR + fR$, where $d = e + f - ef$ taking $e_1 = 1 - f + ef$ we obtain $e = e_1d$ and $e_1R + fR = R$. Since d divides f then any nonunit divisor d' of d cannot be relatively prime to f. To prove that $a = 0$ is an adequate element of R we need to find for any element $b \in R$ some elements $r, s \in R$ such that $0 = rs$, $rR + bR = R$ and if s' is nonunit divisor of s then $s'R + bR = R$. Let f be an idempotent such that $fR = bR$. Then the decomposition

$$R = fR \oplus (1 - f)R$$

implies that $r = 1 - f$, $s = f$ satisfy the desired properties. Theorem is proved. □

Observe that if R is an adequate ring and the adequate condition does not hold when $a = 0$, then R is indecomposable (into the direct product of ring)

Note that any valuation ring trivially satisfies the adequate condition, even for zero element.

Theorem 4.3. *[37, 43] Every nonzero (proper) prime ideal of an adequate Bezout ring R is contained in a unique maximal ideal of R.*

Proof. Suppose that the nozero prime ideal P of R is contained in the intersection of two distinct maximal ideals M_1, M_2 of a ring R. Since M_1, M_2 are distinct maximal ideals, there exist $m_1 \in M_1$, $m_2 \in M_2$ such that

$$m_1R + m_2R = R.$$

Let p be any nonzero element of P. Since R is adequate then there are elements $r, s \in R$ such that $rR + m_1R = R$, $s'R + m_1R \neq R$ for each non unit divisor s' of s. Since P is a prime ideal and $P \subset M_1$ it follows that $s \in P$. Since R is a Bezout ring there is $d \in R$ such that

$$sR + m_2R = dR.$$

Since $P \subset M_2$ then $dR \subset M_2 \neq R$ and d is a nonunit divisor of s. The inclusion

$$dR + m_1R \supset m_2R + m_1R = R,$$

provide us with the contradiction with the adequacy of R. Theorem is proved. □

Example 4.3. **(Henriksen's example.)[ ]** Consider the subring

$$R = \{z_0 + a_1x + a_2x^2 + \dots | z_0 \in \mathbb{Z}, a_i \in \mathbb{Q}\}$$

of the ring of formal power series over $\mathbb{Q}$.

Firstly, we will show that R is a Bezout ring verifying that for each $a, b \in R$, the ideal $aR + bR$ is principal.

The case when a or b equals 0 is trivial, so we assume that neither a nor b is 0. For any nonzero $c \in R$, let $n(c)$ denote the least (nonnegative) integer such that $c_{n(c)} \neq 0$. If we denote $c^* = c_{n(c)}x^{n(c)}$, then then

$$c = c^*(1 + \sum_{k=n(c)+1}^{\infty} \frac{c_k}{c_{n(c)}} x^k).$$

Since the last factor is a unit in R then $cR = c^*R$. By the above

$$aR + bR = a^*R + b^*R.$$

If $n(a) > n(b)$, b^* is a divisor of a^*, so $a^*R + b^*R = b^*R$. Similarly, if $n(a) > n(b)$ then $a^*R + b^*R = a^*R$.

If $n(a) = n(b) = n$ then we can write

$$a^* = \frac{\alpha_1}{\alpha_2}x^n, \quad b^* = \frac{\beta_1}{\beta_2}x^n,$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{Z}$ and $\alpha_2\beta_2 \neq 0$.

If $n = 0$ then take $\alpha_2 = \beta_2 = 1$. Then there exist $\gamma_1, \gamma_2 \in \mathbb{Z}$ such that $\alpha_1\beta_2 = \gamma_1\delta, \quad \alpha_2\beta_1 = \gamma_2\delta$, $(\gamma_1, \gamma_2) = 1$. It can be easily verified that

$$a^*R + b^*R = \frac{\delta x^n}{\alpha_2\beta_2}R.$$

Therefore R is a Bezout ring by definition.

Obviously, $J(R) = \{a \in R | a_0 = 0\}$ and $R/J(R) \cong \mathbb{Z}$.

Hence, $J(R)$ is a prime ideal of R and $\mathbb{Z}$ since contains more than one maximal ideal then using Theorem 4.3 we see that R is not an adequate ring. □

Theorem 4.3 provides a complete description of the spectrum of an adequate ring considering it as a partially order set. Any adequate Bezout ring the set of prime ideals that is contained in a given maximal ideal is linearly ordered since the localization of Bezout ring at any maximal ideal is a valuation ring. Thus if R is an adequate Bezout ring and not an integral domain, then the spectrum of R is disjoint union of totally ordered sets. If R is an adequate Bezout domain, then spectrum of R is a union of totally ordered chains with the only common element (the zero ideal that is prime).

4.2 Stable range of adequate rings

The stable range of ring is one of the important invariants of algebraic K-theory. We will show that the stable range of adequate ring
equals 2.

Definition 4.3. By *the stable range* of ring R we mean the infimum $\mathrm{st.r.}(R)$ of the positive integers n such that whenever

$$a_1R + a_2R + \ldots + a_{n+1}R = R$$

then there are $b_1, \ldots, b_n \in R$ such that

$$(a_1 + b_1a_{n+1})R + \cdots + (a_n + b_na_{n+1})R = R.$$

Definition 4.4. A ring R is called a *ring of stable range 2* if for any $a, b, c \in R$ such that $aR + bR + cR = R$ one can write

$$(a + cx)R + (b + cy)R = R$$

for some elements $x, y \in R$.

Theorem 4.4. *If R is an adequate Bezout ring then* $\mathrm{st.r.}(R) = 2$.

Proof. Let $aR + bR + cR = R$. If $a = 0$, then $bR + cR = R$, and hence

$$(a + c \cdot 1)R + (b + c \cdot 0)R = R$$

and the stable range 2 condition is satisfied. If $a \neq 0$, then $a = rs$, where $rR + bR = R$ and $s'R + bR \neq R$ for any nonunit divisor s' of the element $s \in R$. We claim that

$$(a + c \cdot 0)R + (b + rc)R = aR + (b + rc)R = R.$$

Suppose the contrary, i. e.

$$aR + (b + rc)R = \delta R,$$

where δ is nonunit element of R. Then $aR \subset \delta R$. If $rR + \delta R = hR$, where h is nonunit element of R, then

$$(b + rc)R \subset hR.$$

Since $rR \subset hR$, then $bR \subset hR$ and it is impossible, since $rR + bR = R$. If $sR \subset \delta R \neq R$ then $\delta R + bR = \alpha R$, where α is nonunit element of R. Then

$$(b + rc)R \subset \alpha R$$

and it is impossible, since $aR + bR + cR = R$. Therefore

$$aR + (b + rc)R = R,$$

i.e. $(a + 0 \cdot c)R + (b + rc)R = R$ and $\text{st.r.}(R) = 2$. Theorem is proved. □

Theorem 4.5. *If R is an adequate Bezout ring and $J(R) \neq 0$ then* $\text{st.r.}(R) = 1$.

Proof. Let $bR + cR = R$ and $a \in J(R) \setminus \{0\}$. Then obviously $aR + bR + cR = R$. According to the proof of Theorem 4.4 we have

$$aR + (b + cr)R = R$$

for some element $r \in R$. Since $a \in J(R)$, then $(b + cr)R = R$, i.e. $\text{st.r.}(R) = 2$. Theorem is proved. □

Corollary 4.1. *If R is a Bezout ring and there is an adequate element a in $J(R)$ then* $\text{st.r.}(R) = 1$.

Henriksen studied the following generalization of adequate rings, that have some very nice properties, using the restriction on the cardinality of commutative ring's spectrum.

Definition 4.5. We say the commutative ring R satisfies *Henriksen hypothesis* if for every $a, b \in R$ with $a \notin J(R)$ there is $r \in R$ such that the set of maximal ideals of R containing r is precisely the set of maximal ideals of R containing a and not containing b, i.e.

$$\text{mspec}(r) = \text{mspec}(a) \setminus \text{mspec}(b).$$

Proposition 4.5. *Let R be a commutative adequate ring. Then R satisfies Henriksen hypothesis.*

Proof. Let $a, b \in R$ and $a \notin J(R)$. Since R is an adequate ring then there are some $r, s \in R$ such that $a = rs$, where $rR + bR = R$ and $s'R + bR \neq R$ for any nonunit divisor s' of s. We are going to prove that

$$\text{mspec}(r) = \text{mspec}(a) \setminus \text{mspec}(b).$$

Since $aR \subset rR$ then $\text{mspec}(r) \subset \text{mspec}(a)$. Let $M \in \text{mspec}(b) \cap \text{mspec}(a)$. We claim that $M \in \text{mspec}(r)$. If $M \in \text{mspec}(r)$ then

$$R = rR + bR \subset M.$$

That is impossible. Since $a = rs$, $rR + sR = R$ we have $\text{mspec}(a) = \text{mspec}(r) \cup \text{mspec}(s)$, $\text{mspec}(r) \cap \text{mspec}(s) = \emptyset$. Since $s'R + bR \neq R$ for any nonunit divisor s' of s then we obtain
$mspec(r) = \text{mspec}(a) \setminus \text{mspec}(b)$. Proposition is proved. □

4.3 Zero-adequate and everywhere adequate rings

Now we are going to study commutative rings such that zero is an adequate element in this rings. The structure of such rings allows us to construct more examples of adequate rings.

Theorem 4.6. *Let a be an adequate element of a commutative Bezout ring. Then zero is an adequate element of the quotient-ring R/aR.*

Proof. Let $\overline{b} = b + aR$ be an arbitrary element of the quotient-ring $\overline{R} = R/aR$. If a is an adequate element of R then according to the definition of an adequate element there exist elements $r, s \in R$ such that $a = rs$, where $rR + bR = R$ and $s'R + bR \neq R$ for any nonunit divisor s' of s. Hence

$$\overline{RR} + \overline{b}\overline{R} = \overline{R}.$$

Let $\overline{s} = s + aR$ and $\bar{t}$ be a nonunit divisor $\overline{s} \in \overline{R}$. Then there exist $k \in R$ such that

$$(s + ak)R \subset tR.$$

We claim that $sR + tR \neq R$. Assume the contrary, i.e. $sR + tR = R$. Since $(s + ak)R \subset tR$, then $s + ak = t\beta$ for some $\beta \in R$. Therefore $s(1 + rk) = t\beta$ and the equality $sR + tR = R$ implies that $(1 + rk)R \supset tR$ i.e. $tR + rR = R$. Since $sR + tR = R$ and $tR + rR = R$ then

$$tR+aR=R.$$

The latter means that $\bar{t}=t+aR$ is a unit element of $\overline{R}$ and this is a contradiction to the choice of $\bar{t}$. Thus, we have proved that

$$sR+tR=uR\neq R.$$

Then $uR+bR\neq R$ and hence $\overline{u}\overline{R}+\overline{b}\overline{R}\neq\overline{R}$. Finally we have proved that $\overline{0}$ is an adequate element of $\overline{R}$. Theorem is proved. □

Theorem 4.7. *If zero is an adequate element of commutative Bezout ring R then R is a ring of idempotent stable range 1.*

Proof. Let $bR+cR=R$. Since 0 is an adequate element of the ring R there are $r,s\in R$ then we obtain $0=rs$, where $rR+bR=R$ and $s'R+bR\neq R$ for any nonunit divisor s' of s. We already know that $rR+sR=R$, so

$$ru+sv=1$$

for some $u,v\in R$. Note that the elements $e=ru$ and $1-e=sv$ are idempotents of a ring R. We claim that $(b+ce)R=R$. Suppose

$$(b+ce)R=hR\neq R.$$

Consider $hR+rR=tR$. If t is nonunit element of R then taking into account that

$$(b+ce)R\subset hR\subset tR$$

we obtain that $bR\subset tR$. But this is impossible because $rR\subset tR$, $bR\subset tR$ and $bR+rR=R$. Hence $hR+rR=R$. Moreover we claim that $hR+sR=R$. Suppose $sR+tR=tR\neq R$. Then according to the definition of the element s we obtain that

$$tR+bR=kR\neq R.$$

But on the other hand $(b+ce)R=hR$ and $eR+sR=R$, so $eR+tR=R$ and hence $eR\subset kR$. However, this is impossible as $bR\subset kR$, but $bR+cR=R$. We obtained a contradiction to the assumption and, therefore, $sR+hR=R$. Since $rR+hR=R$ and $sR+hR=R$ then $rsR+hR=R$, i.e $hR=R$. Theorem is proved. □

Since a class of a commutative ring of idempotent stable range 1 coincides with the class of a commutative clean ring due to Proposition 2.17 we derive the following result.

Theorem 4.8. *Let R be a commutative Bezout ring and 0 is an adequate element of R. Then R is a clean ring.*

Naturally, the following question: if R is a commutative Bezout ring and 0 is an adequate element of R, then is R an everywhere adequate ring? In the case of commutative Bezout rings with the finite number of minimal prime ideals we obtain an affirmative answer.

Theorem 4.9. *Let R be a commutative Bezout ring,* 0 *is an adequate element of R and R is a ring with the finite numbers of minimal prime ideals. Then R is an everywhere adequate ring.*

Proof. Let $P_1, P_2, \ldots, P_n$ be all minimal prime ideals of the ring. Denote by

$$N(R) = \bigcap_{i=1}^{n} P_i.$$

Since R is an arithmetical ring then we obtain that $P_i + P_j = R$ for any i, j such that $i \neq j$. According to the Chinese remainder theorem there is a decomposition

$$R/N(R) = R/P_1 \oplus R/P_2 \oplus \ldots \oplus R/P_n.$$

Since every clean ring is a PM-ring then by Theorem 4.8 we conclude that R/P_i are local domains for any $i = 1, 2, \ldots, n$.

Thus there are mutually orthogonal idempotents $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n \in \overline{R}$ such that $\bar{e}_1 + \bar{e}_2 + \ldots + \bar{e}_n = \bar{1}$. Furthermore, we lift them to some mutually orthogonal idempotents **[ ]** $e_1, e_2, \ldots, e_n \in R$. Since $1 - (e_1 + e_2 + \ldots + e_n)$ is an idempotent element in $N(R)$ then it must be zero. Hence

$$R = e_1 R \oplus e_2 R \oplus \ldots \oplus e_n R,$$

where each $e_i R$ is local Bezout domain, i.e. valuation ring **[ ]**. By Theorem 4.11 the ring R is an everywhere adequate ring. Theorem is proved. □

Now we answer another question. Let R be a commutative Bezout ring and zero be an adequate element in the quotient-ring R/aR. Is the element a adequate in R? The answer in the case of a commutative Bezout domain is affirmative and is given by the following theorem.

Theorem 4.10. *Let R be a commutative Bezout domain. If zero is an adequate element of the quotient-ring R/aR then a is an adequate element of the domain R.*

Proof. Let $\overline{R} = R/aR$ and $b \in R$. Since $\overline{0}$ is adequate then there are elements $\overline{R}, \overline{s} \in \overline{R}$ such that

$$\overline{0} = \overline{rs}$$

where $\overline{r}\overline{R} + \overline{b}\overline{R} = \overline{R}$ and $\overline{s}'\overline{R} + \overline{b}\overline{R} \neq \overline{R}$ for any nonunit divisor $\overline{s}'$ of $\overline{s}$ (here $\overline{b} \in \overline{R}$, $\overline{b} = b + aR$).

Since $\overline{r}\overline{R} + \overline{b}\overline{R} = \overline{R}$ then there exist elements $t, u, v \in R$ such that

$$ru + bv = 1 + at$$

(here $\overline{r} = r + aR$). Let $aR + rR = \delta R$. Then $a = \delta a_0$, $r = \delta r_0$ for some elements a_0 and r_0 such that

$$a_0 R + r_0 R = R.$$

Hence $\delta R + bR = R$ and $ru - at + bv = 1$. Since $\overline{0} = \overline{rs}$ then we see that $rs = a\alpha$ for some element $\alpha \in R$ (here $\overline{s} = r + aR$). Hence

$$\delta r_0 s = \delta a_0 \alpha$$

and since R is a domain then $r_0 s = a_0 \alpha$. Since $r_0 R + a_0 R = R$ then $a_0 \beta = s$ for some element $\beta \in R$.

Thus $a = \delta a_0$, where $\delta R + bR = R$, $sR \subset a_0 R$. Then for any nonunit divisor j of a_0 we have $jR + bR \neq R$ i.e. a is an adequate element of a ring R. Theorem is proved. □

In this section we will consider everywhere adequate rings. The first examples of everywhere adequate rings are the von Neumann regular rings and valuation rings. Let's start with the following result.

Proposition 4.6. *Let P be a prime ideal of the ring R that contains at least one adequate element. Then P is contained in a unique maximal ideal of R.*

Proof. Let P be a prime that is not maximal one and let $a \in P$ be an adequate element. Suppose that there are two distinct maximal ideals M_1 and M_2 such that

$$P \subset M_1 \cap M_2.$$

Since $M_1 \neq M_2$ there are elements $m_1 \in M_1$ and $m_2 \in M_2$ such that

$$m_1 + m_2 = 1.$$

Since a is an adequate element we can find elements $r, s \in R$ such that $a = rs$, where $rR + m_1 R = R$ and for any nonunit divisor s' of the element s we have

$s'R + m_1R \neq R$. Since P is a prime ideal then $s \in P$. Let $sR + m_2R = dR$. Since $s \subset M_2$ and $m_2 \in M_2$ then d is nonunit divisor of the element m_2. Hence we have $dR + m_1R \neq R$. But

$$dR + m_1R \supset m_2R + m_1R = R$$

that contradicts the adequacy of element a. □

Proposition 4.7. *A commutative Bezout ring with the unique minimal prime ideal is everywhere adequate if and only if it is a valuation ring.*

Proof. The necessity of this statement follows from Proposition 4.6. On the other hand, the sufficiency part follows from fact that Bezout ring with unique minimal prime ideal is a ring with unique maximal ideal, i. e. it is a valuation ring. □

Theorem 4.11. *A commutative Bezout ring with a finite number of minimal prime ideals is everywhere adequate ring if and only if it is a finite direct sum of valuation rings.*

Proof. Let $P_1, \dots, P_n$ be all minimal prime ideals of R and $N(R)$ be a nilradical of R. Since every Bezout ring is arithmetical then the ideals P_i are pairwise comaximal, and hence $R/N(R)$ is a direct sum of domains $R/P_1, \dots, R/P_n$ by Chinese remainder theorem.

Thus there exist mutually orthogonal idempotents $\overline{e_1}, \dots, \overline{e_n}$ where $\overline{e_i} \in R/P_i$ such that

$$\overline{e_1} + \ldots + \overline{e_n} = \overline{1}.$$

Then we lift them to mutually orthogonal idempotents $e_1, \dots, e_n \in R$ **[ ]**. Since

$$1 - (e_1 + \ldots + e_n)$$

is an idempotent element in $N(R)$ then it must be zero. Thus

$$R = e_1R \oplus \ldots \oplus e_nR$$

and each e_iR is a minimal prime ideal.

Since every $e_iR \cong R/(1 - e_i)R$ is homomorphic image of R then R is a Bezout ring and by Proposition 4.6 each e_iR is a valuation ring.

Let R be the direct sum of valuation rings $R_i = e_iR$. It easy to see that R is a Bezout ring. Let $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ be the elements of R with $a \neq 0$. Define the elements $r, s \in R$ coordinatewise in the following way:

$$r_i = \begin{cases} 1, & \text{if } b_i \text{ is nonunit in } R_i, \\ a_i, & \text{if } b_i \text{ is unit in } R_i; \end{cases}$$

$$s_i = \begin{cases} a_i, \text{ if } b_i \text{ is nonunit in } R_i, \\ \quad 1, \text{ if } b_i \text{ is unit in } R_i. \end{cases}$$

Clearly $a = rs$ and $rR + bR = R$. Let $s' = (s'_1, \ldots, s'_n)$ be a nonunit of R which divides s; if s'_i is a nonunit, then s_i is a nonunit and hence b_i is a nonunit. Thus $s'R_i + b_iR_i \neq R_i$. Therefore $s'R + bR \neq R$.

Since a valuation ring is any everywhere adequate ring then each R_i is adequate and the adequacy condition holds for $a = 0$ in each R_i. Therefore R is everywhere adequate ring. Theorem is proved. □

Theorem 4.12. *Everywhere adequate Bezout ring is a clean ring.*

Proof. Let R be an everywhere adequate ring and $b \in R$. Let $0 = rs$, where $rR + bR = R$ and $s'R + bR \neq R$ for each nonunit divisor s' of the element s. We claim that $sR + rR = R$. If we suppose that

$$sR + rR = \delta R \neq R$$

then $\delta R + bR = R$since δ is a as divisor of r and $\delta R + bR \neq R$ as δ is a divisor of s. Suppose that $sR + (1-b)R = \delta R \neq R$, then $s = \delta s_0$ and let $\delta R + bR = hR \neq R$. Then $b = hb_0$ and $1 - b = hy$ for some elements $b_0, y \in R$. It means that $hR = R$. Therefore

$$sR + (1-b)R = R.$$

Since $rR + sR = R$ then we see that $ru + sv = 1$ for some elements $u, v \in R$. Since $rs = 0$, we have $r^2u = r$, $s^2v = s$. If we denote $e = ru$, then $e^2 = e$ and $1 - e = sv$. Since $rR + bR = R$ we obtain that

$$r\alpha + b\beta = 1$$

for some elements $\alpha, \beta \in R$. Here $svb\beta = sv$, i.e. $1 - e \in bR$. Similarly, $e \in (1-b)R$ so R is an exchange ring and by Proposition 2.17 R is a clean ring. Theorem is proved. □

Since any clean ring is a PM-ring then according to Theorem 4.12 and Proposition 2.17 we obtain

Theorem 4.13. *Everywhere adequate Bezout ring is a PM-ring.*

Since any clean ring is a ring of idempotent stable range 1 then by Theorem 4.12 we have a next result.

Theorem 4.14. *Everywhere adequate Bezout ring is a ring of idempotent stable range 1.*