Chapter 5
Finite homomorphic images

5.1 Minimal prime spectrum and fractionally regular rings

In this section, we are going to investigate the influence of commutative Bezout ring's spectrum on the matrix diagonal reduction possibility. One of the following results is: a commutative Bezout ring with finitely many minimal prime ideals is an elementary divisor ring if and only if any quotient-ring of R with respect to prime ideals is an elementary divisor ring.

For an arbitrary element x of the ring R we denote by

$$D(x) = \{P \in \min R | x \notin P\}$$

the basic set of the Zariski topology on $\min R$. We say that $\min R$ is a compact if $\min R$ is a compact in this topology.

Proposition 5.1. *An Hermite ring R is an elementary divisor ring if and only if $R/N(R)$ is an elementary divisor ring.*

Proof. The necessity follows from the fact that any homomorphic image of an elementary divisor ring is an elementary divisor ring. So, it suffices to consider only the case when $R/N(R)$ is an elementary divisor ring. Due to Theorem 3.6 in order to prove that R is an elementary divisor ring, it suffices to show that for any $a, b, c \in R$ such that

$$aR + bR + cR = R$$

there exist elements $p, q \in R$ such that

$$(ap + bq)R + cqR = R.$$

Since $R/N(R)$ is an elementary divisor ring then by Theorem 3.6 for elements $\overline{a}, \overline{b}, \overline{c} \in R/N(R)$ there exists elements $\overline{p}, \overline{q}, \overline{u}, \overline{v} \in R/N(R)$ such that

$$(\overline{ap}+\overline{bq})\overline{u}+\overline{cqv}=\overline{1}$$

($\overline{a},\overline{b},\overline{c}$ are the homomorphic images of the elements a,b and c under the canonical map of R onto $R/N(R)$).

Hence, it is obvious that there exist elements $p,q,u,v \in R$ and $n \in N(R)$) such that

$$(ap+bq)u+cqv=1+n.$$

Since $1+n \in U(R)$ then

$$(ap+bq)R+cqR=R,$$

that completes the proof. The Proposition is proved. □

Theorem 5.1. *A Bezout ring with finitely many minimal prime ideals is an elementary divisor ring if and only if for an arbitrary prime ideal, the quotient ring with respect to this ideal is an elementary divisor ring.*

Proof. Since any homomorphic image of an elementary divisor ring is an elementary divisor ring then we need to prove only sufficiency part. In view of Theorem 4.11 we have that

$$R/N(R) \cong R/P_1 \oplus R/P_2 \oplus \ldots \oplus R/P_n$$

is a direct sum of elementary divisor rings and, therefore, $R/N(R)$ is an elementary divisor ring. We establish that R is an Hermite ring. Hence, R is an elementary divisor ring. Theorem is proved. □

Now let's consider the Bezout rings R such that the quotient rings $Q_{Cl}(R)$ contain only finitely many minimal prime ideals. Curious reader can check that any ring whose quotient rings with respect to the nilradical are Goldie rings satisfies the mentioned condition.

Theorem 5.2. *Let R be a Bezout ring such that its classical quotient ring $Q_{Cl}(R)$ has finitely many minimal prime ideals. Then*

$$R=R_1 \oplus R_2 \oplus \ldots \oplus R_n,$$

where $R_1,R_2,\ldots,R_n$ are Bezout ring with finitely many minimal prime ideal.

Proof. By **[ ]** there is a decomposition

$$Q_{Cl}(R)=e_1Q_{Cl}(R) \oplus e_2Q_{Cl}(R) \oplus \ldots \oplus e_nQ_{Cl}(R),$$

where $e_1Q_{Cl}(R), e_2Q_{Cl}(R), \ldots, e_nQ_{Cl}(R)$ are Bezout ring with unique minimal prime ideal.

Let $S = e_1R \oplus \ldots \oplus e_nR$ since R is a distributive ring then all idempotents of $Q_{Cl}(R)$ are in R ([ ], Lemma 1.10). Hence, we have $S = R$. It is obvious that if P is a minimal prime ideal in $e_iQ_{Cl}(R)$, then $P \cap R$ is a minimal prime ideal in R that is contained in e_iR, i.e., the ring e_iR contains a unique minimal prime ideal. Theorem is proved. □

Note that the ring e_iR (see Theorem 5.2) is an Hermite ring **[ ]** and as an obvious consequence of Theorem 5.2 we obtain the following statement:

Corollary 5.1. *Let R be a Bezout ring such that $R/N(R)$ is a Goldie ring. Then R is an Hermite ring.*

Theorem 5.3. *Let R be a Bezout ring in which an arbitrary minimal prime ideal is contained in unique maximal ideal and its classical quotient ring is a ring with finitely many minimal prime ideals. Then R is an Hermite ring and, moreover, is a finite direct sum of valuation rings.*

Proof. By Theorem 5.2 we have a decomposition

$$R = e_1R \oplus \ldots \oplus e_nR,$$

where each e_iR contains a unique minimal prime ideal, which is, obviously, an ideal in R. Hence e_iR is a local Bezout ring and consequently a valuation ring, which is obviously an Hermite ring. Using Theorems 4.11 and 4.4 we finish the proof of theorem. □

Theorem 5.4. *Let R be a Bezout ring such that its arbitrary minimal prime ideal is contained in a unique maximal ideal and the classical quotient ring of R has finitely many minimal prime ideals. Then R is an elementary divisor ring.*

Let R be a Bezout ring whose quotient ring $R/N(R)$ is a Goldie ring. In the view of Corollary 5.1 R is an Hermite ring. Furthermore, $Q_{Cl}(R/N(R))$ is a Bezout ring with finitely many minimal prime ideals. Assume that an arbitrary prime ideal of R is contained in unique maximal ideal. By Theorem 5.4 $R/N(R)$ is an elementary divisor ring and using Proposition 5.1 we obtain that R is an elementary divisor ring. Consequently we obtain the following statement:

Theorem 5.5. *Let R be a Bezout ring such that its arbitrary minimal prime ideal is contained in a unique maximal ideal and the quotient ring by the nil-radical is a Goldie ring. Then R is an elementary divisor ring.*

Now we consider the case of adequate rings.

Theorem 5.6. *Every Hermite ring whose classical quotient ring is a Boolean ring is an elementary divisor ring.*

Proof. Let R be an Hermite ring such that $Q_{Cl}(R)$ is a Boolean ring. By Proposition 5.1 we can assume that R is a ring without nilpotent elements. Let P be an arbitrary minimal prime ideal of R. Then for any $a \in P$ an element $1/a \in Q_{Cl}(R)$ is an idempotent and, consequently the element a is also an idempotent. Thus we have shown that the arbitrary element of any minimal prime ideal of R is an idempotent.

To prove the theorem it suffices to show that the matrix

$$A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix},$$

where $aR + bR + cR = R$, admits canonical diagonal reduction. First, we consider the case when a is not an idempotent. Hence for arbitrary idempotent $e \in R$ the elements $a(1-e)$ and ae are idempotents. Therefore

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \begin{pmatrix} 1-e & -e \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a(1-e) & -ae \\ x & y \end{pmatrix},$$

where $(a(1-e))^2 = a(1-e)$ and $(ae)^2 = ae$. If we denote $a(1-e) = \phi$ and $ae = f$ and assume that $d = \phi + f - \phi f$, then

$$\begin{pmatrix} \phi & f \\ x & y \end{pmatrix} \begin{pmatrix} 1 & -f \\ 1-\phi & \phi_1 \end{pmatrix} = \begin{pmatrix} d & 0 \\ \alpha & \beta \end{pmatrix},$$

where $d^2 = d$ and $dR + \alpha R + \beta R = R$. We set $r = 1 - d + d\beta$ and $s = d + \beta - d\beta$. As a result, we get $\beta = rs$, $d = ds$. Thus,

$$\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & 0 \\ \alpha & \beta \end{pmatrix} = \begin{pmatrix} d + r\alpha & r\beta \\ \alpha & \beta \end{pmatrix} = B.$$

It is obvious that $(d + r\alpha)R + r\beta R = R$. Since R is an Hermite ring we establish that the matrix B and, consequently, the matrix A admits canonical diagonal reduction. If a is an idempotent, then using the reasoning presented above we complete the proof of theorem. □

Theorem 5.7. *Let R be a reduced Bezout ring that is a Goldie ring. Then an arbitrary minimal prime ideal of R is principal and is generated by an idempotent.*

Proof. The restrictions on R, imply that the classical quotient ring $Q_{Cl}(R)$ is an artinian von Neumann regular ring with finitely many minimal prime ideals. Let P be a minimal prime ideal of the ring R. Consider the ideal

$$P_Q = \{p/s \mid p \in P\}.$$

It is obvious that P_Q is a prime ideal of $Q_{Cl}(R)$. Thus, there exists an idempotent $e \in Q_{Cl}(R)$ such that

$$P_Q = eQ_{Cl}(R).$$

Since R is a distributive ring then we have $e \in R$. For any $p \in P$ we obtain that $p = er$, where r is a von Neumann regular element. Hence,

$$ep = eer = er = p$$

and $P \subset eR$. But $e \in P$ so we get $eR \subset P$ and $P = eR$. Theorem is proved. □

The purpose of this section is to determine the structure of a commutative Bezout ring in which for every nonzero and nonunit element $a \in R$ the classical quotient ring $Q(R/J(aR))$ is a von Neumann regular, where $J(aR)$ is the Jacobson radical of R/aR.

Definition 5.1. A ring R is called *an almost Baer ring* if for each $x \in R$ there exists an element $y \in R$ such that $\mathrm{Ann}(xR) = yR$.

Theorem 5.8. *Let R be a reduced almost Baer ring. Then $Q(R)/J(Q(R))$ is a von Neumann regular ring.*

Proof. Let P be a prime ideal of R that is not a minimal one. By **[ ]** there exists an element $x \in P$ such that $\mathrm{Ann}(x) \not\subset P$. Let $t \in \mathrm{Ann}(xR + yR)$, where $yR = \mathrm{Ann}(xR)$. Then $t(xu - yv) = 0$ for each $u, v \in R$. Hence $txu = tyv$, where $(txu)^2 = txu \cdot tyv = xuvt^2 \cdot y = 0$ as $y \in \mathrm{Ann}(xR)$. Since R is reduced then $txu = tyv = 0$ for each $u, v \in R$. Therefore $t \in \mathrm{Ann}(xR) = yR$ and $t = yw$ for some $w \in R$. Then $t^2 = tyw = 0$ due to the proved property of t. Since R is a reduced ring then $t = 0$ and $\mathrm{Ann}(xR + yR) = 0$. Hence by **[ ]** $x - y$ is a not a zero divisor in R. Therefore $Q(R)$ is a ring in which every prime ideal is maximal, i.e. $Q(R)$ is zero-dimensional By **[ ]** $Q(R)/J(Q(R))$ is a von Neumann regular ring. Theorem is proved. □

Theorem 5.9. *Let R is the Bezout domain and $a \in R \setminus \{0\}$, then R/aR is an almost Baer ring.*

Proof. Suppose $b \in R$ and $bR \subset aR$. Then $(b : aR) = \{r \in R \mid rb \in aR\} = sR$, where $a = bs$, so $(s : aR) = bR$. Thus the annihilator of any principal ideal in R/aR is principal. The Theorem is proved. □

Theorem 5.10. *Let R be a fractionally regular Bezout ring. Then R is a ring of stable range 3.*

Proof. Let $a,b,c,d \in R$ and $aR+bR+cR+dR = R$ and $\overline{R} = R/J(aR)$. Since $Q(\overline{R})$ is a von Neumann regular ring and st.r. = st.r.$(\overline{R})$, we have by Lemma **[ ]** that st.r.$(\overline{R}) = 2$. Since $\overline{b}\overline{R}+\overline{c}\overline{R}+\overline{d}\overline{R} = \overline{R}$ and st.r.$(\overline{R}) = 2$ then $(\overline{b}+\overline{d}\overline{x})\overline{R}+(\overline{c}+\overline{d}\overline{y})\overline{R} = \overline{R}$ for some $\overline{x},\overline{y} \in \overline{R}$. We claim that

$$(a+d\cdot 0)R+(b+dx)R+(c+dy)R = R.$$

If not then there would be a maximal ideal M for which $a, b+dx, c+dy \in M$. But this is impossible since this would imply that $M/J(aR)$ is a maximal ideal of $\overline{R}$ containing $\overline{b}+\overline{d}\overline{x}$ and $\overline{c}+\overline{d}\overline{y}$. Consequently $(a+d\cdot 0)R+(b+dx)R+(c+dy)R = R$. Thus we have shown that R has stable range 3. The Theorem is proved. □

Theorem 5.11. [ ] *Let R be a Bezout PM-ring such that* min$R$ *is compact. Then* st.r.$(R) = 1$.

Theorem 5.12. *Let R be a fractionally regular Bezout PM-ring. Then R is an elementary divisor ring.*

Proof. Let $\overline{R} = R/J(aR)$ for nonzero and nonunit element $a \in R$. By Theorem 5.11 $\overline{R}$ has stable range 1. Let $aR+bR+cR = R$. Since $\overline{b}\overline{R}+\overline{c}\overline{R} = R$ and st.r.$(\overline{R}) = 1$ there is $y \in R$ such that

$$(\overline{b+cy})\overline{R} = \overline{R}.$$

We claim that $(a+c\cdot 0)R+(b+cy)R = R$. In opposite case there would be some maximal ideal M such that $a, b+yc \in M$. But this is impossible as this would imply that $M/J(aR)$ is a maximal ideal of $\overline{R}$ containing an invertible element $\overline{b+yc}$. Consequently $(a+0\cdot c)R+(b+yc)R = R$. We have shown that R has stable range 2 and by Theorem 3.2 R is an Hermite ring.

Furthermore, it is proved that if $aR+bR+cR = R$ then there exists an element $y \in R$, such that $aR+(b+cy)R = R$. By Theorem 3.6 R is elementary divisor ring. Theorem is proved. □

Lemma 5.1. *Let R be the Bezout ring, then the following statements are equivalent.*

1. R is semihereditary ;

2. min$R$ *is compact and R is reduced.*

Theorem 5.13. *Every fractionally regular Bezout ring of a stable range 2 is an elementary divisor ring.*

Proof. By [43] it is sufficient to prove that a module M named by the matrix

$$A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix},$$

where $aR+bR+cR=R$ is a direct sum of cyclic R-modules. By **[ ]** is easy to check that M is an R/acR - module. Let $J=J(acR)$. Since R is fractionally regular then $\overline{R}=R/J$ is semihereditary. Thus, $\overline{M}=M/J(M)$ is the module named by the matrix

$$\overline{A} = \begin{pmatrix} \overline{a} & \overline{0} \\ \overline{b} & \overline{c} \end{pmatrix}.$$

Since $\overline{R}$ is semihereditary an $\overline{ac}=\overline{0}$ there is an idempotent $\overline{e}\in\overline{R}$ such that $\overline{a}=\overline{e}a$ and $\overline{b}=(\overline{1}-\overline{e})\overline{b}$. By **[ ]** it is easily seen that $\overline{M}\overline{e}$ is named by $\overline{ea}$ as an $\overline{e}\overline{R}$ - module and $\overline{M}(\overline{1}-\overline{e})$ is named by $(\overline{1}-\overline{e})\overline{a}$ as $(\overline{1}-\overline{e})\overline{R}$-module. The rings $\overline{e}$ and $(\overline{1}-\overline{e})\overline{R}$ are the homomorphic images of $\overline{R}$, so are Hermite and there exist the invertible matrices $\overline{P_1}\in M_2(\overline{e}\overline{R})$ and $Q_1\in M_2((\overline{1}-\overline{e})\overline{R})$ such that

$$\overline{P_1}\,\overline{ea} = \begin{pmatrix} \overline{s} & \overline{0} \\ \overline{0} & \overline{0} \end{pmatrix}, (\overline{1}-\overline{e})\overline{a}\overline{Q_1} = \begin{pmatrix} \overline{0} & \overline{0} \\ \overline{0} & \overline{t} \end{pmatrix}.$$

If we denote by $\overline{P}=(\overline{1}-\overline{e})\overline{e}+\overline{P_1}$ and $\overline{Q}=\overline{ee}+\overline{P_1}$, where E is 2 by 2 identity matrix over $\overline{R}$ then $\overline{P}$, $\overline{Q}$ are invertible matrices over $\overline{R}$ such that

$$\overline{PAQ} = \begin{pmatrix} \overline{s} & 0 \\ 0 & \overline{t} \end{pmatrix}.$$

By **[ ]** we may assume that $\overline{s}$ divides $\overline{t}$. It is obviously, that $\overline{R}=\overline{a}\overline{R}+\overline{b}\overline{R}+\overline{c}\overline{R}=\overline{s}\overline{R}$ and $\overline{s}$ is a unit of $\overline{R}$. It follows that $\overline{M}$ is a cyclic $\overline{R}$-module. By Nakayama lemma it follows that M is also cyclic over R/acR. Hence M is cyclic over R. It follows that $M\cong R/tR$ by **[ ]** so every finitely presented R - module is a direct sum of cyclic modules. By **[ ]** R is an elementary divisor ring. Theorem is proved. □

Theorem 5.14. *Let R be a Bezout ring of stable range 2 of right Krull dimension. Then R is an elementary divisor ring.*

Proof. Let $a\in R\setminus\{0\}$, $a\notin U(R)$ and $\overline{R}=R/J(aR)$. Then $\overline{R}$ is an Hermite reduced ring with right Krull dimension. By **[ ]**, $\overline{R}$ is an S-ring, and every principal ideal of $\overline{R}$ is projective. By **[ ]**, R is a fractionally regular Hermite ring. By Theorem 5.13, R is an elementary divisor ring. Theorem is proved.

Theorem 5.15. *Let R be a Bezout ring of stable range 2 with Noetherian spectrum. Then R is an elementary divisor ring.*

Proof. Since every ideal of R contains only finitely many minimal prime ideals **[ ]**, then for every $a \in R \setminus \{0\}$ the ring $R/J(aR)$ is a semiherieditary ring and R is a fractionally regular Bezout ring of a stable range 2 **[ ]**. By Theorem 5.14 R is an elementary divisor ring. Theorem is proved. □

5.2 Semiregularity of zero-adequate rings

In this section we will prove that for a commutative Bezout ring the semiregularity condition is precisely the same as the adequacy requirement of its zero element. We start with a few results concerning the Jacobson radical of such rings.

Theorem 5.16. *Let R be a commutative ring such that zero is an adequate element of R. Then* $\overline{0} = 0 + J(R)$ *is an adequate element of the quotient-ring* $R/J(R)$.

Proof. Let $\overline{R} = R/J(R)$ and $\overline{b} = b + J(R)$ be any element of $\overline{R}$. Then $0 = rs$, where $rR + bR = R$ and $s'R + bR \neq R$ for any nonunit divisor s' of s. Let $\overline{R} = r + J(R)$, $\overline{s} = s + J(R)$. Therefore $\overline{R}\overline{R} + \overline{b}\overline{R} = \overline{R}$, $\overline{s}'\overline{R} + \overline{b}\overline{R} \neq \overline{R}$. Let $\overline{t}$ be nonunit divisor of $\overline{s} \in \overline{R}$. Then there exist $j \in J(R)$ and $k \in R$ such that $s + j = tk$. If $sR + tR = R$ then $su + tu = 1$ for some elements $u, v \in R$. Hence $tku - ju + tv = 1$ and $tku + tv = 1 + ju$. Since $j \in J(R)$ then $tku + tv$ is unit element of R and hence t is unit element of R that contradict the choice of $\overline{t} \in \overline{R}$.

Therefore $\overline{t}$ is a nonunit element of R and

$$sR + tR = kR \neq R.$$

Since k is a nonunit divisor of s then $kR + bR \neq R$. Inasmuch $\overline{k}\overline{R} + \overline{b}\overline{R} \neq \overline{R}$ and $\overline{t}\overline{R} + \overline{b}\overline{R} \neq \overline{R}$ then $\overline{0}$ is an adequate element of $\overline{R}$. Theorem is proved. □

In the following theorem we describe the elements of commutative Bezout ring where zero is an adequate element via of Jacobson radical.

Theorem 5.17. *Let R be a commutative Bezout ring in which zero is an adequate element. Then for any nonunit element* $b \in R$ *there exist idempotent* $e \in R$ *such that* $be \in J(R)$ *and* $eR + bR = R$.

Proof. By the adequacy of zero element there are $s, r \in R$ such that $0 = rs$, where $rR + bR = R$, and $s'R + bR \neq R$ for any nonunit divisor s' of s. As we shown many times before

$$rR + sR = R.$$

Hence $ru + sv = 1$ for some elements $u, v \in R$ and obviously $r^2u = r$, $s^2u = s$. If we denote by $e = ru$ then $e^2 = e$, $1 - e = sv$ and $eR + bR = R$.

Since $eR + bR = R$ then $rux + by = 1$ for some elements $x, y \in R$ and hence $svrux + svby = sv$, i.e $1 - e \in bR$. Therefore we obtain an inclusion $\mathrm{mspec}(b) \subset \mathrm{mspec}(1 - b)$.

Suppose there is $M \in \mathrm{mspec}(1-b)$ and $b \notin M$. Hence $M + bR = R$, i.e. $m + bt = 1$ for some elements $m \in M$, $t \in R$. Let

$$(1 - e)R + mR = dR.$$

Since $1 - e \in M$, $m \in M$ then $d \in M$. Recall that $sv = 1 - e$ and d is nonunit divisor of sv. Since $s^2v = s$ then d is also a nonunit divisor of s. Then $bR + dR \neq R$. But $R = bR + mR \subseteq bR + dR \neq R$ that is impossible. Therefore $\mathrm{mspec}(b) = \mathrm{mspec}(1 - b)$.

Since $\mathrm{mspec}(e) \cup \mathrm{mspec}(1 - e) = \mathrm{mspec}(R)$ then for the arbitrary maximal ideal M of R there are only two possibilities:

1. $M \in \mathrm{mspec}(e)$;
2. $M \in \mathrm{mspec}(1 - e)$.

If $M \in \mathrm{mspec}(e)$, then $be \in M$. Otherwise $M \in \mathrm{mspec}(1-e) = \mathrm{mspec}(b)$, so $b \in M$ and hence $be \in M$. As a result we obtain that $be \in J(R)$ as was desired. □

Theorem 5.18. *Zero element is an adequate in a commutative semiprime Bezout ring R if and only if R is a von Neumann regular ring.*

Proof. By Theorem 5.17 for any nonunit element $b \in R$ there exists an idempotent $e \in R$ such that

$$be \in J(R), \quad eR + bR = R.$$

Since $J(R) = 0$ then $be = 0$ and the equality $eR + bR = R$ implies that

$$bu + ev = 1$$

for some elements $u, v \in R$. Hence $b^2u = b$, i.e. b is a von Neumann regular element. Theorem 4.2 completes the proof. Theorem is proved. □

By Theorems 5.16 and 5.18 we have the following result.

Theorem 5.19. *Let R be a commutative Bezout ring. The following statements are equivalent:*

1. zero is an adequate element of R;

2. R is a semiregular ring.

Using Theorem 5.19 we obtain:

Theorem 5.20. *A commutative Bezout domain R is an adequate domain if and only if R/aR is a semiregular ring for any nonzero element $a \in R$.*

5.3 Avoidable rings

A commutative adequate Bezout domain can be determined as a commutative Bezout ring in which every finite homomorphic images are semiregular ring. What kind a commutative Bezout domain in which every finite homomorphic images are clean (exchange) ring?

Definition 5.2. A commutative ring is said to be an *avoidable ring* if for any elements $a,b,c \in R$, $c \neq 0$ such that $aR + bR + cR = R$ there exist elements $r,s \in R$ such that $c = rs$, where $rR + aR = R$, $sR + bR = R$ and $rR + sR = R$.

Theorem 5.21. *Let R be a commutative ring. If R is an avoidable ring then for any $c \in R \setminus \{0\}$ the quotient ring R/cR is a clean ring. If R is a Bezout domain and for any $c \in R \setminus \{0\}$ the factor ring R/cR is a clean ring then R is an avoidable ring.*

Proof. Let R be a commutative Bezout domain. Denote by $\overline{R} = R/cR$ and $\overline{a} = a + cR$, $\overline{b} = b + cR$. Since $\overline{R}$ is a clean ring, then there exists an idempotent $\overline{e} \in \overline{R}$ such that

$$\overline{e} \in \overline{a}\overline{R}, \quad (\overline{1} - \overline{e}) \in \overline{b}\overline{R}.$$

Since $\overline{e} \in \overline{a}\overline{R}$ than $e - ap = cs$ for some elements $p, s \in R$. Similarly

$$1 - e + b\alpha = c\beta$$

for some elements $\alpha, \beta \in R$. After the substitution of $e = cs + ap$ into $1 - e = b\alpha + c\beta$ we will get

$$apR + bR + cR = R.$$

Since $\overline{e} = \overline{e}^2$ then $e(1-e) = ct$ for some element $t \in R$. Let $eR + cR = dR$. Since R is a commutative Bezout domain we have

$$e = de_0, \quad c = dc_0,$$

where $e_0R + c_0R = R$. As $e(1-e) = ct$ and $e_0R + c_0R = R$ we have $e + c_0 j = 1$ for some element $j \in R$. Taking $r = c_0$, $s = d$ we obtain a decomposition $c = rs$ where $rR + eR = R$ and $sR \subset eR$. Since $e = ap + cs$ then

$$rR + apR = R, \quad sR \subset apR.$$

If $sR \subset apR$ then $sR + bR = R$. Obviously $rR + sR = R$ and $rR + aR = R$.

For the converse statement let R be an avoidable ring, $aR + bR + cR = R$ and $c = rs$, where $rR + sR = R$, $rR + aR = R$ and $sR + bR = R$. Denote

$$\overline{R} = r + cR, \quad \overline{s} = s + cR.$$

Since $rR + sR = R$ then one has $ru + sv = 1$ and $\overline{R}^2\overline{u} = \overline{R}$, $\overline{s}^2\overline{v} = \overline{s}$ for some $u, v \in R$.

Let $\overline{sv} = \overline{e}$. Then clearly $\overline{e}^2 = \overline{e}$ and $\overline{1} - \overline{e} = \overline{R}\overline{u}$. Since $rR + aR = R$ then $\overline{a}\overline{\beta}\overline{e} = \overline{e}$ for some element $\overline{\beta} \in \overline{R}$.

Similarly $\overline{b}\overline{x}(\overline{1} - \overline{e}) = \overline{1} - \overline{e}$ for some element $\overline{x} \in \overline{R}$. Therefore, we have proved that if $\overline{a}\overline{R} + \overline{b}\overline{R} = \overline{R}$ then there exists an idempotent $\overline{e} \in \overline{R}$ such that

$$\overline{e} \in \overline{a}\overline{R}, \quad (\overline{1} - \overline{e}) \in \overline{b}\overline{R}.$$

Hence $\overline{R}$ is an exchange ring and by Proposition 2.17 $\overline{R}$ is also a clear one. The theorem is proved. □

Theorem 5.22. *Every commutative adequate Bezout domain R is an avoidable ring.*

Proof. Let $aR + bR + cR = R$ and $c \neq 0$. Since R is an adequate domain then $c = rs$, where $rR + aR = R$ and $s'R + aR \neq R$ for any nonunit divisor $s' \in R$ of element s. Obviously $rR + sR = R$ (see proof of Theorem 4.12).

Let $sR + bR = dR \neq R$. Since d is nonunit divisor of an element s then $dR + aR = hR \neq R$ by the adequacy of $a \in R$. As $cR \subset dR \subset hR$, $bR \subset dR \subset hR$, $aR \subset hR$, then we have that $aR + bR + cR \subset hR \neq R$. It is impossible because $aR + bR + cR = R$. Theorem is proved. □

By the straightforward consequence of Theorem 5.21 and the fact that a clean ring ia a *PM*-ring we have the following result.

Theorem 5.23. *An avoidable domain is a domain in which every nonzero prime ideal is contained in a unique maximal ideal.*

Similarly to the definition of avoidable ring we can define an avoidable element.

Definition 5.3. An element a of a commutative ring R is said to be an *avoidable element* if for any elements $b,c \in R$ such that $aR+bR+cR=R$ there exist elements $r,s \in R$ such that $a=rs$, where $rR+bR=R$, $sR+cR=R$ and $rR+sR=R$.

According to the proof of Theorem 5.22 we have a next result.

Proposition 5.2. *Any adequate element of a commutative Bezout domain is an avoidable element.*

Theorem 5.24. *Let a be an avoidable element of a commutative Bezout domain. Then zero is an avoidable element of the quotient-ring R/aR.*

Proof. Let $\overline{R}=R/aR$ and

$$\overline{b}\overline{R}+\overline{c}\overline{R}=\overline{R},$$

where $\overline{b}=b+aR$, $\overline{c}=c+aR$. By the assumption there are $r,s \in R$ such that $a=rs$, where $rR+bR=R$, $sR+cR=R$ and $rR+sR=R$. Taking the images in the quotient ring $\overline{R}$ we obtain $\overline{0}=\overline{R}\overline{s}$, where $\overline{RR}+\overline{b}\overline{R}=\overline{R}$, $\overline{s}\overline{R}+\overline{c}\overline{R}=\overline{R}$ and $\overline{RR}+\overline{s}\overline{R}=\overline{R}$. □

Theorem 5.25. *A commutative ring R is clean if and only if* 0 *is an avoidable element.*

Proof. Let $0=rs$ and $bR+cR=R$ where $rR+bR=R$, $sR+cR=R$ and $rR+sR=R$. Since $rR+sR=R$, then

$$ru+sv=1$$

for some elements $u,v \in R$. Since $0=rs$, then $r^2u=r$, $s^2v=s$. If we denote $e=ru$ then $e^2=e$ and $1-e=sv$. Since $rR+bR=R$ then

$$r\alpha+b\beta=1$$

for some elements $\alpha,\beta \in R$.

Therefore, $svb\beta=sv$, i.e. $1-e \in bR$, and similarly $e \in cR$. Hence R is an exchange ring. Since R is a commutative ring then by Proposition 2.17 R is a clean ring. The necessity is proved.

For the sufficiency we need to show that in any clean ring zero is an avoidable element. Let $bR+cR=R$. The commutative clean ring is the same as an exchange one due to Proposition 2.17, so R is an exchange ring. Then there exists an idempotent $e \in R$ such that $e \in bR$ and $(1-e) \in cR$. Since $0=e(1-e)$ we obtain

$$(1-e)R+bR=R,$$

$$eR+cR=R \text{ and } eR+(1-e)R=R.$$

If we take $r=1-e$, $s=e$ then we obtain a desired representation of zero element. Theorem is proved. □

Theorem 5.26. *Let R be a commutative Bezout domain. If $\overline{0}$ is an avoidable element of R/aR then a is an avoidable element of R.*

Proof. Let $\overline{R} = R/aR$. Since zero is an avoidable element of $\overline{R}$ then for any elements $\overline{b}, \overline{c} \in \overline{R}$ such that $\overline{b}\overline{R} + \overline{c}\overline{R} = \overline{R}$ there exist $r, s \in R$ such that $\overline{0} = \overline{R}\overline{s}$, where $\overline{R}\overline{R} + \overline{b}\overline{R} = \overline{R}\,\overline{s}\overline{R} + \overline{c}\overline{R} = \overline{R}$ and $\overline{R}\overline{R} + \overline{s}\overline{R} = \overline{R}$. Let $\overline{R} = r + aR$, $\overline{s} = s + aR$, $\overline{b} = b + aR$, $\overline{c} = c + aR$. Then $\overline{b}\overline{R} + \overline{c}\overline{R} = \overline{R}$ implies that $aR + bR + cR = R$. As $\overline{R}\overline{R} + \overline{b}\overline{R} = \overline{R}$ then are elements $u, v, t \in R$ such that

$$ru + bv = 1 + at.$$

Let $aR + rR = \delta R$. Then $a = \delta a_0$, $r = \delta r_0$ and $a_0 R + r_0 R = R$ for some elements $a_0, r_0 \in R$. Since $ru + bv = 1 + at$ then we obtain $\delta R + bR = R$.

Inasmuch $\overline{0} = \overline{R}\overline{s}$ we obtain $rs = a\alpha$ for some element $\alpha \in R$. Then $\delta r_0 s = \delta a_0 \alpha$. As R is a domain then $r_0 s = a_0 \alpha$. Since $r_0 R + a_0 R = R$ then there are elements $t, k \in R$ such that

$$r_0 t + a_0 k = 1.$$

The latter means that $a_0 \beta = s$ for some element $\beta \in R$. Therefore $a = \delta a_0$ where $\delta R + bR = R$ and $sR \subset a_0 R$. Since $\overline{s}\overline{R} + \overline{c}\overline{R} = \overline{R}$ then $a_0 R + cR = R$. Finally, $\overline{R}\overline{R} + \overline{s}\overline{R} = \overline{R}$ implies that $\delta R + a_0 R = R$. Thus we have shown that a is an avoidable element. Theorem is proved. □

As an obvious consequence we obtain the following result.

Theorem 5.27. *Let R be a commutative Bezout domain such that for any nonzero element $a \in R$ zero element of R/aR is avoidable. Then R is an avoidable ring.*

5.4 Effective and Dirichlet rings

Among the commutative Bezout domains whose finite homomorphic images are semipotent (or a commutative Bezout domains whose every nonzero element is semipotent) we introduse a new class of rings and call them effective. Furthermore, we will prove that effective rings are elementary divisor rings.

Definition 5.4. A commutative Bezout domain is said to be *effective* if for any elements $a, b, c \in R$ such that $aR + bR + cR = R$ and $cR + bR \neq R$ there exists an element $p \in R$ such that ${}_a\mathrm{A}_{pc}$ and $pR + bR + aR = R$

An obvious example of effective domain is any adequate ring. Henriksen's example (see Example 4.3) $R = \{z_0 + a_1x + a_2x^2 + \ldots | z_0 \in \mathbb{Z}, a_i \in \mathbb{Q}\}$ is an effective ring that is not adequate. Notice that in Henriksen example given the equality $aR + bR = R$ it follows that a or b is an adequate element.

Theorem 5.28. *Every effective domain is an elementary divisor ring.*

Proof. Let $aR + bR + cR = R$. Let $aR + bR = R$ and

$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}.$$

To prove this theorem it is enough to show that the matrix A admits the canonical diagonal reduction. The cases $aR + bR = R$, $bR + cR = R$ and $aR + cR = R$ are obvious. By the definition of an effective ring there is some element $q \in R$ such that ${}_a\mathrm{A}_{cq}$ and $aR + bR + cqR = R$. Since $a = ps$ where $pR + cqR = R$ and $s'R + cqR \neq R$ for any $s' \in R$ such that $sR \subset s'R \neq R$. Let's show that $apR + (bq + cq)R = R$. Suppose the contrary, that is $apR + (bq + cq)R = hR \neq R$. Since $ap = p^2s$, then let $pR + hR = dR \neq R$. As $pR \subset dR$ and $hR \subset dR$ we have $cqR \subset hR$ that is impossible since $pR + cqR = R$. Thus $sR \subset hR$. By the definition of element s we have that $hR + cqR = kR \neq R$. Let $kR + cR = xR \neq R$. Then $bpR \subset xR$ and $aR \subset xR$, $cR \subset xR$ and $aR + bR + cR = R$ then $xR + bR = R$. So $pR \subset xR$ that is impossible as $pR + sR = R$. As a result we have $qR \subset kR$. Then $bpR \subset kR$. If $bR + kR = \alpha R \neq R$ then $aR \subset \alpha R$, $bR \subset \alpha R$ and $qR \subset \alpha R$. Using the fact that $aR + bR + qR = R$ we obtain a contradiction, so α must be a unit. Hence $pR \subset kR$, but this is also impossible as $pR + sR = R$ and $sR \subset kR$. So, we have proved that $apR + (bp + cq)R = R$. By Theorem 3.6 we have shown that A admits the canonical diagonal reduction. Theorem is proved. □

In the previous chapter we have studied neat rings as rings whose homomorphic images are exchange ones. We'll show that in the case of commutative Bezout rings neat rings are effective. To complete this purpose we need the following proposition.

Proposition 5.3. *A commutative ring R is an exchange ring if and only if for any pair of elements $a, b \in R$ such that $aR + bR = R$ there is an idempotent $e \in R$ such that $e \in aR$ and $(1 - e) \in bR$.*

Proof. By **[ ]** in every exchange ring equality $aR + bR = R$ implies that there are orthogonal idempotents e and $1 - e$ that $e \in aR$ and $(1 - e) \in bR$. The sufficiency is obvious. Proposition is proved. □

Theorem 5.29. *Let R be a commutative Bezout domain and for any quotient ring $a \in R \setminus \{0\}$ R/aR is an exchange ring. Then R is an effective ring.*

Proof. Let $\overline{R} = R/aR$ be an exchange ring for any $a \in R \setminus \{0\}$. By Proposition 5.3 the equality $\overline{c}\overline{R} + \overline{b}\overline{R} = \overline{R}$ implies that we can find an idempotent $\overline{e} \in \overline{R}$, such that $\overline{e} \in \overline{c}\overline{R}$ and $\overline{1} - \overline{e} \in \overline{b}\overline{R}$. Let's notice that the equality $\overline{c}\overline{R} + \overline{b}\overline{R} = \overline{R}$ implies, that $aR + bR + cR = R$. Since $\overline{e} \in \overline{a}\overline{R}$ then there is some element $p \in R$ such that $e - cp = as$ for some element $s \in R$. Similarly $1 - e - b\alpha = a\beta$ for some elements $\alpha, \beta \in R$. Substituting $e = as + cp$ in $1 - e - b\alpha = a\beta$ we'll get $cp + aw + b\alpha = 1$ for some element $w \in R$. The latter means that $pR + aR + bR = R$. Let's prove that ${}_a\mathrm{A}_{cp}$. Since $\overline{e} = \overline{e}^2$ then $e(1-e) = at$ for some element $t \in R$. If we consider an ideal $eR + aR = dR$, we have $e = de_0$ and $a = da_0$ for some elements $e_0, c_0 \in R$ such that $e_0R + a_0R = R$. Then $e_0(1-e) = a_0t$, and then $e + c_0\gamma = 1$, for some element $\gamma \in R$. Taking $r = a_0$, $s = d$ we obtain a decomposition $a = rs$, where $rR + eR = R$ and $sR \subset eR$. So we have ${}_a\mathrm{A}_e$. Since $e = as + cp$ then by Proposition 5.28 we have ${}_a\mathrm{A}_{cp}$. Theorem is proved. □

Proposition 5.4. *Let R be a commutative Bezout domain, which for any elements $a, b, c \in R$ that $aR + bR + cR = R$ there exists an element $p \in R$ such that ${}_aA_{cp}$. Then $pR + aR + bR = R$ and $a = rs$ where $rR + cpR = R$ and $s'R + apR \neq R$, for any nonunit divisor s' of element s if and only if $sR + bR = R$.*

Proof. Let $a = rs$ where $rR + pR = R$ and $s'R + cpR \neq R$ for any nonunit divisor s' of element s. If $pR + aR + bR = R$ then $cpR + aR + bR = R$. If $sR + bR = \delta R \neq R$ then $\delta R + cpR = hR \neq R$. It is impossible, since $cpR + aR + bR = R$.

Suppose $sR + bR = R$. We need to prove that $cpR + bR + aR = R$. If $pR + bR + aR = hR \neq R$, then $pR \subset hR$, $aR \subset hR$. Since $rR + cpR = R$ then h is a nonunit divisor of s. By $bR \subset hR$ and $sR \subset hR$ we have that $sR + bR \subset hR \neq R$. It is impossible since $sR + bR = R$. Proposition is proved. □

Theorem 5.30. *Let R be a commutative Bezout domain in which for any elements $a, b, c \in R$ such that $aR + bR + cR = R$ there exists such element $p \in R$ that ${}_aA_{cp}$ and $pR + bR + aR = R$. Then R/aR is an exchange ring.*

Proof. Let $\overline{R} = R/aR$ and $\overline{c}\overline{R} + \overline{b}\overline{R} = \overline{R}$, where $\overline{c} = c + aR$, $\overline{b} = b + aR$. Then $aR + bR + cR = R$ and there exists an element $p \in R$ such that $a = rs$, where $rR + cpR = R$ and $s'R + cpR \neq R$ for each nonunit divisor $s'R$ of s, and $pR + bR + aR = R$. Obviously $ru + sv = 1$ for some element $u, v \in R$.

Hance $\overline{R}^2\overline{u} = \overline{R}$, $\overline{s}^2\overline{v} = \overline{s}$ and $\overline{e} = \overline{sv}$, $\overline{e}^2 = \overline{e}$, $\overline{R}\overline{u} = \overline{1} - \overline{e}$ and $\overline{s}\overline{R} = \overline{e}\overline{R}$, $(\overline{1} - \overline{e})\overline{R} = \overline{RR}$. By Proposition 5.4, we obtain $sR + bR = R$ as we have $\overline{RR} \subset \overline{b}\overline{R}$.

Since $rR+cpR=R$ then $\overline{s}\overline{R}\subset\overline{c}\overline{R}$, i.e. we have $\overline{e}\overline{R}\subset\overline{b}\overline{R}$ and $(\overline{1}-\overline{e})\overline{R}\subset\overline{c}\overline{R}$. By Proposition 5.29 R/aR is an exchange ring. Theorem is proved. □

5.5 Neat range one

As it was proved in the previous section a commutative Hermite ring is precisely a Bezout ring of stable range 2. There is a similar description of commutative elementary divisor rings.

Definition 5.5. A commutative ring R is said to be a *ring of neat range 1* if for any $a,b,c\in R$ such that $aR+bR=R$ and for any $c\in R\setminus\{0\}$ there exists $u,v,t\in R$ such that $a+bt=uv$, where $uR+cR=R$, $vR+(1-c)R=R$ and $uR+vR=R$.

An obvious example of a ring of neat range 1 is any ring of stable range 1.

Proposition 5.5. *Let R be a commutative Bezout ring and let a is an element of R such that for any $c\in R$ there are $u,v\in R$ such that $a=uv$ where $uR+cR=R$, $vR+(1-c)R=R$ and $uR+vR=R$. Then R/aR is a clean ring.*

Proof. Let's introduce notations $\overline{R}=R/aR$, $\overline{u}=u+aR$, $\overline{v}=v+aR$. Since $uR+vR=R$, then $ru+sv=1$ and $\overline{R}\overline{u}^2=\overline{u}$, $\overline{s}\overline{v}^2=\overline{v}$, where $\overline{R}=r+aR$, for some $r,s\in R$.

Let $\overline{sv}=\overline{e}$, obviously $\overline{e}^2=\overline{e}$, $\overline{s}=s+aR$ and $\overline{1}-\overline{e}=\overline{R}\overline{u}$. Since $uR+cR=R$ we have $\overline{csv}\overline{\beta}=\overline{sv}$ for some element $\overline{\beta}\in\overline{R}$. Similarly

$$(\overline{1}-\overline{c})\overline{ru}\overline{\alpha}=\overline{ru}$$

for some element $\overline{\alpha}\in\overline{R}$. Therefore, for any element $\overline{c}\in\overline{R}$ there exists an idempotent $\overline{e}\in\overline{R}$ such that

$$\overline{e}\in\overline{c}\overline{R},\quad \overline{1}-\overline{e}\in(\overline{1}-\overline{c})\overline{R},$$

i.e. $\overline{R}$ is an exchange ring. Since $\overline{R}$ is a commutative ring then $\overline{R}$ is a clean ring by Proposition 2.17. Proposition is proved. □

Proposition 5.6. *Let R be an Hermite ring and for any coprime triple $a,b,c\in R$ there exist $p,q\in R$ such that $paR+(pb+qc)R=R$. Then there exist elements $t,u,v\in R$ such that $b+tc=uv$, where $uR+aR=R$ and $vR+cR=R$.*

Proof. Since $paR+(pb+qc)R=R$, then $pR+(pb+qc)R=R$. The latter implies that $pR+cR=R$ that is $pv+cj=1$ for some $v,j\in R$. Therefore, $v(pb+qc)=b+ct$ for some $t\in R$. Moreover $vR+cR=R$ and if we denote $u=pb+qc$ then $uR+aR=R$. Proposition is proved. □

Proposition 5.7. *Let R be a commutative Hermite ring and for any $a,b,c \in R$ such that $aR+bR+cR=R$ there exists $t \in R$ such that $b+tc=uv$ and $vR+cR=R$, $uR+aR=R$. Then there are $p,q \in R$ such that $aR+(pb+qc)R=R$, $pR+cR=R$.*

Proof. Since $vR+cR=R$ then $pv+cj=1$ for some $p,j \in R$. Therefore

$$pb=p(uv-tc)=puv-ptc=u-cq$$

for some $q \in R$. Hence $(pb+qc)R+aR=R$, $pR+cR=R$. Proposition is proved. □

Theorem 5.31. *A commutative Hermite ring R is an elementary divisor ring if and only if R is a ring of neat range 1.*

Proof. Let $aR+bR+cR=R$. Since R is an Hermite ring then $bR+cR=dR$, by the Theorem 3.1 we have $b=b_1d$, $c=c_1d$ where $b_1R+c_1R=R$. Since $aR+bR+cR=R$ then $aR+dR=R$. Hence

$$ax+dd_1=1$$

for some $x,d_1 \in R$. Therefore $1-dd_1 \in aR$. Since R is a ring of neat range 1, then $b_1+tc_1=u_1r$ where

$$u_1R+(1-dd_1)R=R, \;\; rR+dd_1R=R.$$

Let $u=u_1d$. Since $1-dd_1 \in aR$ and $u_1R+aR=R$ then $uR+aR=R$. Consider the expression

$$b+tc=(b_1+tc_1)u=ru.$$

Since $vR+dd_1R=R$ then $vR+dR=R$. Hence

$$vR+cR=vR+c_1dR=vR+c_1R.$$

Moreover, as $b_1+tc_1=uv_1$ then $vR+c_1R=R$ and consequently $vR+cR=R$. By Propositions 5.5 and 5.6 we obtain that R is an elementary divisor ring.

Conversely, let R be an elementary divisor ring and let $xR+yR=R$ and $z \in R \setminus \{0\}$. By Proposition 5.5 taking $a=z$, $b=x$, $c=(1-z)y$ we have

$$zR+xR+(1-z)yR=aR+bR+cR.$$

Hence $x+t(1-z)y=vu$, where $vR+y(1-z)R=R$ and $uR+zR=R$. If $\mu=1-z$ then $x+\mu y=ru$ where

$$vR+(1-z)R=R, \;\; uR+zR=R.$$

Note that the elements u and v can be chosen such that $uR+vR=R$, and finally we have proved that R is a ring of neat range 1. Theorem is proved. □

The following fact establishes the connection between the different ranges defined for the ring R. Especially, we will derive the connection between the introduced neat range 1 and the stable range of the ring.

5.6 Meaningful ring

Here we will investigate commutative Bezout domains whose finite homomorphic images are semipotent rings.

Let's start our research with a few simple useful properties of adequacy.

Proposition 5.8. *Let R be a commutative Bezout domain and $a \in R$ is some nonzero element. Then ${}_aA_b$ if and only if ${}_aA_{b+at}$ for any element $t \in R$.*

Proof. Assume a is a adequate to the element b in R. By definition there are some elements $r, s \in R$ such that $a = rs$ where $rR + bR = R$ as well as $s'R + bR \neq R$ for any nonunit divisor s' of s. Taking arbitrary $t \in R$ we consider the ideal

$$rR + (b + at)R = hR.$$

As $aR \subset rR \subset hR$ and $(b + at)R \subset hR$ then $bR \subset hR$ that is impossible because $rR + bR = R$. Hence $h \in U(R)$.

Since $s'R + bR \neq R$ for any nonunit divisor s' of s then

$$s'R + (b + at)R \neq R$$

too. Thus we obtained that ${}_a\mathrm{A}_{b+at}$ for any element $t \in R$ and necessity is proved.

For the converse statement suppose that ${}_a\mathrm{A}_{b+at}$, that means that one can find some elements $r, s \in R$ such that $a = rs$ where $rR + (b + at)R = R$, $s'R + (b + at)R \neq R$ for some nonunit divisor s' of s.

If $rR + bR = hR \neq R$ for some element h then $aR \subset rR \subset hR$ and $bR \subset hR$ implies $(b + at)R$ that is impossible because $rR \subset hR$.

Next assume that there is some element $s' \in R$ such that $sR \subset s'R \neq R$, $s'R + bR = R$. Then $s'R + (b + at)R = hR \neq R$ and $hR \subset bR$ that contradicts the assumption $s'R + bR = R$. Proposition is proved. □

Proposition 5.9. *Let R be a commutative Bezout domain and ${}_aA_b$. Then element $\overline{b} = b + aR$ is a clean element in $\overline{R} = R/aR$.*

Proof. Since a is adequate to the element b in a Bezout domain R then there are some elements $r, s \in R$ such that $a = rs$ where $rR + bR = R$ and $s'R + bR \neq R$ for any nonunit divisor s' of s. Going down to factor-ring $\overline{R}$ we have

$$\overline{R}\overline{R} + \overline{b}\overline{R} = \overline{R}$$

and

$$\overline{s}'\overline{R} + \overline{b}\overline{R} \neq \overline{R}$$

where $\overline{R} = r + aR$, $\overline{s}' = s' + aR$. Really if you have $\overline{R}\overline{R} + \overline{b}\overline{R} = \overline{R}$ then

$$s' - b\beta = 1 + a\alpha$$

for some elements $\alpha, \beta \in R$. Since $aR \subset s'R$ we have $s'R + bR = R$. This is impossible since s' is a nonunit divisor of s.

Let $\bar{t}$ be some nonunit divisor of $\overline{s} \in \overline{R}$, where $\bar{t} = t + aR$. Then there is some element $k \in R$ such that

$$(s + ak)R \subset tR.$$

Let's show that $sR + tR \neq R$. Suppose the contrary, that is $sR + tR = R$. Since

$$(s + ak)R \subset tR,$$

then

$$s + ak = t\beta$$

for some element $\beta \in R$. Equality $s + rsk = t\beta$ implies $s(1 + rk) = t\beta$. As $sR + tR = R$ then

$$(1 + rk)R \subset tR,$$

so $rr + tR = R$. Since $tR + sR = R$ and $tR + rR = R$ we have $tR + rsR = R$, i.e., $tR + aR = R$. From here $\bar{t}\overline{R} = \overline{R}$ it is impossible since $\bar{t}$ is a nonunit divisor $\overline{s} \in \overline{R}$. Thus we have proved that $sR + tR = uR$ and $uR \neq R$. This means

$$\bar{t}\overline{R} + \overline{b}\overline{R} \neq \overline{R}.$$

So we obtained that $\overline{0} = \overline{R}\overline{s}$ is an adequate element for the element $\overline{b}$, i.e., ${}_{\overline{0}}\mathrm{A}_{\overline{b}}$. Obviously $\overline{R}\overline{R} + \overline{s}\overline{R} = \overline{R}$, i.e., $\overline{R}\overline{u} + \overline{s}\overline{v} = \overline{1}$ for some element $\overline{u}, \overline{v} \in \overline{R}$.

Additionally, we have

$$(\overline{R}\overline{u})^2 = \overline{R}\overline{u},$$

and

$$(\overline{s}\overline{v})^2 = \overline{s}\overline{v}.$$

Let denote $\overline{e} = \overline{R}\overline{u}$. Now we want to prove that $\overline{b} - \overline{e}$ is a unit in $\overline{R}$. Suppose that

$$(\overline{b} - \overline{e})\overline{R} = \overline{h}\overline{R} \neq \overline{R}.$$

Consider an ideal

$$\overline{h}\overline{R}+\overline{RR}=\overline{t}\overline{R}.$$

If $\overline{t}$ is nonunit in $\overline{R}$ then

$$(\overline{b}-\overline{e})\overline{R}\subset\overline{h}\overline{R}\subset\overline{t}\overline{R}.$$

Since $\overline{RR}=\overline{e}\overline{R}$, we have

$$\overline{b}\overline{R}\subset\overline{t}\overline{R}.$$

This is impossible since

$$\overline{RR}\subset\overline{h}\overline{R},$$

and $\overline{b}\overline{R}\subset\overline{h}\overline{R}$ but $\overline{b}\overline{R}+\overline{RR}=\overline{R}$. So $\overline{h}\overline{R}+\overline{RR}=\overline{R}$.

Now we will prove that

$$\overline{h}\overline{R}+\overline{s}\overline{R}=\overline{R}.$$

Assume that

$$\overline{h}\overline{R}+\overline{s}\overline{R}=\overline{t}\overline{R}\neq\overline{R}.$$

Since $\overline{t}$ is a nonunit divisor of $\overline{s}$ we have

$$\overline{t}\overline{R}+\overline{b}\overline{R}=\overline{k}\overline{R}\neq\overline{R}.$$

On the other hand

$$(\overline{b}-\overline{e})\overline{R}=\overline{h}\overline{R}$$

and

$$\overline{e}\overline{R}+\overline{s}\overline{R}=\overline{R},$$

so we have that $\overline{e}\overline{R}+\overline{t}\overline{R}=\overline{R}$ implies $\overline{e}\overline{R}\subset\overline{k}\overline{R}$. Last inclusion is impossible since $\overline{b}\overline{R}\subset\overline{k}\overline{R}$. At last we have that $\overline{s}\overline{R}+\overline{h}\overline{R}=\overline{R}$. Since

$$\overline{RR}+\overline{h}\overline{R}=\overline{R}$$

and

$$\overline{R}\overline{s}\overline{R}+\overline{h}\overline{R}=\overline{R}.$$

As we know $\overline{0}=\overline{R}\overline{s}$, then $\overline{h}$ is a unit of the ring $\overline{R}$. So we have proved that $\overline{b}-\overline{e}=\overline{u}$ is a unit in a ring $\overline{R}$ and hence $\overline{b}=\overline{e}+\overline{u}$ is a clean element. Proposition is proved. □

We note that $\overline{c}=c+aR$ is a unit element of R/aR if and only if $cR+aR=R$. Really if $cR+aR=R$ then $cu+av=1$ for some elements $u,v\in R$. Then $\overline{cu}+\overline{av}=\overline{1}$ and $\overline{cu}=\overline{1}$. Obviously if $\overline{cx}=\overline{1}$ for some element $\overline{x}=x+aR$ we have $cx=1+ay$ for some element $y\in R$. Here from $cR+aR=R$.

By the Proposition 5.9 we have if ${}_a\mathrm{A}_b$ then $b = ru + c + at$ where $cR + aR = R$ and t is a some element of R. And where $a = rs$, $rR + bR = R$ and $s'R + bR \neq R$ for some nonunit divisor s' of s and $ru + sv = 1$.

Since $ruR + bR = R$ we obtain the following criterion.

Proposition 5.10.

Definition 5.6. [ ] An element $a \in R$ is said to be *an exchange element* if there exists idempotent $e \in R$ such that $e \in aR$ and $1 - e \in (1 - a)R$.

Definition 5.7. [ ] An element $a \in R$ is said to be *an element of idempotent stable range 1* if $aR + bR = R$ and there exists idempotent $e \in R$ such that $a + be$ is a unit element of R for any element $b \in R$.

By **[ ]** and Proposition 5.9 we have a next result.

Theorem 5.32. *Let R be a commutative Bezout domain and ${}_a\mathrm{A}_b$. Then*

1. $\overline{b} = b + aR$ is a clean element of R/aR;

2. $\overline{b} = b + aR$ is an exchange element of R/aR;

3. $\overline{b} = b + aR$ is element of idempotent stable range 1 of R/aR.

Put the following questions:

Question 5.1. Let R be a commutative Bezout domain and $a \in R \setminus \{0\}$. Let $\overline{b} = b + aR \in R/aR$ is a clean element. Whether ${}_a\mathrm{A}_b$?

The answer on this question is the following result.

Definition 5.8. Let R be a commutative Bezout domain. Say that an element $a \in R$ is an avoidable for $b \in R$ if $a = rs$ where $rR + bR = R$ and $sR + (1 - b)R = R$ and $rR + sR = R$. This fact is denoted ${}_a\mathrm{R}_b$.

Proposition 5.11. *Let R be a commutative Bezout domain. If ${}_a\mathrm{A}_b$ then ${}_a\mathrm{R}_b$.*

Proof. Let $a = rs$ where $rR + bR = R$ and $s'R + bR \neq R$ for some nonunit divisor s' of s. Obviously we have that $rR + sR = R$. Show that $sR + (1 - b)R = R$. Let $sR + (1 - b)R = hR \neq R$. Since h is nonunit divisor of s we have $hR + bR = \delta R \neq R$. Let $h = \delta h_0$, $b = \delta b_0$. Since $sR + (1 - b)R = hR$ we have $su + (1 - bv = h$ for some $u, v \in R$ and $s = \delta s_0$, $1 - b = ht$ for some $s_0, t \in R$. Then $sR \subset \delta R$ and $1 - b \subset \delta R$, i.e., $\delta R + bR = R$. This is impossible since $sR \subset \delta R \neq R$ and $\delta R + bR \neq R$. Therefore ${}_a\mathrm{R}_b$. Proposition is proved. □

Theorem 5.33. *Let R be a commutative Bezout domain. Then ${}_aR_b$ if and only if an element $\overline{b} = b + aR$ is exchange (clean, idempotent stable range 1) element in R/aR.*

Proof. Let ${}_a\mathrm{R}_b$ then $a = r_0 s$ where $rR+bR = R$, $sR+(1-b)R = R$ and $rR+sR = R$. Since $rR+sR = R$, then $ru+sv = 1$ for some $u, v \in R$. Since $\overline{R}\overline{s} = \overline{0}$ we have $\overline{R}^2\overline{u} = \overline{R}$, $\overline{s}^2\overline{v} = \overline{s}$ and $\overline{R}\overline{u} = \overline{e}$, $\overline{su} = 1-e$ where $\overline{e}^2 = \overline{e}$. Since $rR+bR = R$ we have $rx+by = 1$ for some element $x, y \in R$. Then $\overline{sv}\overline{by} = \overline{sv}$, i.e., $(\overline{1}-\overline{e})\overline{by} = \overline{1}-\overline{e}$, $\overline{1}-\overline{e} \in \overline{b}\overline{R}$. Similar to that shown $\overline{e} \in (\overline{1}-\overline{b})\overline{R}$, i.e., $\overline{b}$ is an exchange element and hence $\overline{b}$ is a clean (idempotent stable range 1) element.

Let $\overline{b} = b + aR$ is an exchange element in R/aR. Then there exist idempotents $\overline{e}$ such that $\overline{e} \in \overline{b}\overline{R}$ and $\overline{1}-\overline{e} \in (\overline{1}-\overline{b})\overline{R}$. Since $\overline{e} \in \overline{b}\overline{R}$ one has that $e - bp = as$ for some elements $p, s \in R$. Similarly $1-e-(1-b)\alpha = a\beta$ for some elements $\alpha, \beta \in R$.

Since $\overline{e}^2 = \overline{e}$ we obtain $e(1-e) = at$ for some element $t \in R$. Let $eR+aR = dR$. We have $e = de_0$, $a = da_0$ where $e_0R+a_0R = R$. Since $e(1-e) = at$ and R is a domain. We have $e_0(1-e) = a_0t$.

Since $e_0R+a_0R = R$ we have $e+a_0j$ for some element $j \in R$. Taking $r = a_0$, $s = d$ we obtain the decomposition $a = rs$ where $rR+eR = R$ and $rR \subset eR$. Since $e = ap+as$ we have $rR+bpR = R$, $sR \subset bpR \subset bR$. Then $sR+(1-b)R = R$ and $rR+bR = R$ and $rR+sR = R$, i.e., ${}_a\mathrm{R}_b$. Theorem is proved. □

Return to our sheep. We know, that if R is a commutative Bezout domain and:

1. if $a \in R \setminus \{0\}$ and for any $b \in R$ there is a decomposition $a = rs$, where $rR+bR = R$, $sR+(1-b)R = R$ and $rR+sR = R$ then R/aR is an exchange (clean, idempotent stable range 1) ring;
2. if for any $a \in R \setminus \{0\}$ and $b \in R$ there is a decomposition $a = rs$, where $rR+bR = R$, $sR+(1-b)R = R$ then R is a domain in which every nonzero prime ideal is contained in unique maximal ideal.

As in the mentioned results the mentioned results the properties of R/aR depends on the type of the decomposition $a = rs$ we naturally ask the following:

Question 5.2. Let R be a commutative Bezout domain and $a \in R \setminus \{0\}$ is such that for any $b \in R$ there is a decomposition $a = rs$, where $rR+bR = R$ and $rR+sR = R$. What one can prove about R/aR?

The following result answers this question.

Theorem 5.34. *Let R be a commutative Bezout domain, $a \in R \setminus \{0\}$ and let for any $b \in R$ there is a decomposition $a = rs$, where $rR + bR = R$ and $rR + sR = R$. Then $\overline{R} = R/aR$ is a semipotent ring and the converse is also true.*

Proof. The equality $rR + sR = R$ implies that $ru + sv = 1$ for some element $u, v \in R$ and $\overline{R}^2\overline{u} = \overline{R}$ and $\overline{s}^2\overline{v} = \overline{s}$. Obviously $\overline{Ru} = \overline{e}$ and $\overline{e}^2 = \overline{e}$ and $\overline{1} - \overline{e} = \overline{sv}$. Similarly as $rR + bR = R$ then $\overline{1} - \overline{e} \in \overline{bR}$. Since $\overline{1} - \overline{e}$ is an idempotent, then $\overline{R}$ is a semipotent ring. Note that if $b \notin J(aR)$ then $\overline{1} - \overline{e}$ is a proper idempotent. Really, if for any maximal M that contains an element a we have $b \in M$ then r is a unit element, and the converse is also true.

Let for $\overline{b} = b + aR$ we have that there exists an idempotent $\overline{e} = \overline{e}^2$ such that $\overline{e} \in \overline{bR}$. Since $\overline{e}^2 = \overline{e}$ then $e(1-e) = a\alpha$ for some $\alpha \in R$. As $\overline{e} \in \overline{bR}$ then $e - bt = as$ for some elements $t, s \in R$. Let $eR + aR = dR$ then $e = de_0$, $a = da_0$ and $a_0R + e_0R = R$. Factoring out element d from the equality $e(1-e) = a\alpha$ we obtain $e_0(1-e) = a_0\alpha$. Also, as $a_0R + e_0R = R$ then $aR + e_0R = R$. Taking $r = a_0$, $s = d$ we get that $rR + bR = R$ (since $e - bt = as$) and $rR + sR = R$. Theorem is proved. □

Definition 5.9. Let R be a commutative Bezout domain. An element $a \in R \setminus \{0\}$ is said to be *a semipotent element* in R if R/aR is a semipotent ring.

Since an exchange ring is a semipotent ring then every avoidable element is obviously, a semipotent one.

In the following result we establish the connection concepts of adequate elements and semipotent ones.

Theorem 5.35. *Let R be a commutative Bezout domain. $a \in R$ is a semipotent element then for any element $b \notin J(aR)$ there exists some element $u \in R$ that ${}_a\mathrm{A}_{bu}$.*

Proof. Let $\overline{R} = R/aR$ be a semipotent element and $b \notin J(aR)$. By the semipotency of $\overline{R}$ for $\overline{b} = b + aR$ there is some nonzero idempotent $\overline{e}$ such that $\overline{e}^2 \in \overline{bR}$. Hence we can find elements $u, t \in R$ such that $e - bu = at$. Moreover, as $\overline{e}^2 = \overline{e}$ then $e(1-e) = as$ for some element $s \in R$. Let $eR + aR = dR$, where $e = de_0$, $a = da_0$ and $e_0R + a_0R = R$. Cancellating 2 we get $e_0(1-e) = a_0s$ and $e + a_0j = 1$ for some element $j \in R$. Taking $r = a_0$, $s = d$ we obtain a decomposition $a = rs$, where $rR + eR = R$ and $s'R + eR \neq R$ for some nonunit divisor s' of s.

Thus ${}_a\mathrm{A}_e$ and by Proposition 5.8 the equality $bu = e + at$ implies ${}_a\mathrm{A}_{bu}$. Theorem is proved. □

It is worth to note that in general if ${}_a\mathrm{A}_{bc}$ in a commutative Bezout domain R then it is not true that ${}_a\mathrm{A}_b$ or ${}_a\mathrm{A}_c$. As a corollary of previous theorems we obtain the following result.

Theorem 5.36. *Let R be a commutative Bezout domain. An element $a \in R$ is semipotent if and only if for any element $b \notin J(aR)$ there exists some element $u \in R$ such that ${}_aA_{bu}$, $bu \notin aR$.*

5.7 Bezout morphic rings and units lifting

Proposition 5.12. *If R is a commutative Bezout domain and $0 \neq a \in R$, then R/aR is a morphic ring.*

Proof. Suppose $b \in R$ and $Ra \subseteq Rb$. Then

$$(b : a) = \{r \in R \mid rb \in Ra\} = Rs,$$

where $a = bs$, so $(b : a) = Ra$. Thus the annihilator of any principal ideal of R/Ra is principal ideal. And we also shows that every principal ideal of R/Ra is the annihilator of principal ideal. Moreover, if $I_1 = \mathrm{Ann}(J_1)$, $I_2 = \mathrm{Ann}(J_2)$ with I_i, J_i principal ideals we have

$$\mathrm{Ann}(I_1 \cap I_2) = \mathrm{Ann}(\mathrm{Ann}(J_1) \cap \mathrm{Ann}(J_2)) = \mathrm{Ann}(\mathrm{Ann}(J_1 + J_2)) = J_1 + J_2 = \mathrm{Ann}(J_1) + \mathrm{Ann}(J_2).$$

By Lemma 2.2 we have, that R/Ra is morphic ring. □

Theorem 5.37. *Let R be a commutative Bezout domain. Then for any nonzero element $a \in R$ we have that R/aR is a morphic ring.*

As a consequence of this fact we can give an example of a commutative morphic ring that is not clean. It is a negative answer to a question of Nicholson in [49].

Example 5.1. Let R be Henriksen's ring from Example 4.3:

$$R = \{z_0 + a_1x + a_1x^2 + \ldots | z_0 \in \mathbb{Z}, a_i \in \mathbb{Q}, i = 1, 2, \ldots\}.$$

We have shown that R is a commutative Bezout domain [**?**]. The quotient-ring R/xR according to Theorem 5.12 is a morphic ring but it is not clean since a homomorphic image of the ideal $N = \{a_1x + a_1x^2 + \ldots | a_i \in \mathbb{Q}, i = 1, 2, \ldots\}$ is an ideal N/xR that is prime, but belongs to all maximal ideals in the quotient-ring R/xR. That is why R/xR is not clean because any clean ring has to be a PM-ring. Note that $xR \neq N$ as $x/2 \in N$ but $x/2 \notin xR$.

Definition 5.10. A ring R is called a *left (right) Kasch ring* if every simple left (right) R-module embeds in ${}_RR$ (R_R) that is $r(L) \neq 0$ $(l(L) \neq 0)$ for every (maximal) left (right) ideal L in a ring R.

Theorem 5.38. *Let R be a commutative Bezout domain R and a is some nonzero element in R. The following statements are equivalent:*

1. R/aR is a Kasch ring;

2. Any maximal ideal M that contains the element a is principal.

Proof. $(1) \Rightarrow (2)$. Consider a Kasch ring R/aR and let $\overline{M}$ be a maximal ideal in this ring. We can write $\mathrm{Ann}(\overline{M}) = \overline{H}$ where $\overline{H}$ is an ideal in R/aR and $\overline{H} \neq \{\overline{0}\}$. Since $\overline{H}$ annihilates the maximal ideal $\overline{M}$ then we can write $\overline{HM} = \{\overline{0}\}$. Since the maximal ideal $\overline{M}$ belongs to $\mathrm{Ann}(\overline{H})$ then by maximality of $\overline{M}$ we have that $\overline{M} = \mathrm{Ann}(\overline{H}) \neq \overline{R}$.

Since $\overline{M}$ is a maximal ideal then for every element $\overline{d} \neq \overline{0}$ which belongs to the ideal $\overline{H}$ we have the equality $\overline{d}\overline{M} = \{\overline{0}\}$. Thus we obtain that the maximal ideal $\overline{M}$ belongs to $\mathrm{Ann}(\overline{d})$, where $\overline{d}$ is a nonunit.

Hence $\overline{M} = \mathrm{Ann}(\overline{d}) = \overline{b}\overline{R}$ because R/aR is a morphic ring. Therefore, $\overline{M} = \overline{b}\overline{R}$ and $M = bR + aR = cR$, because R is a commutative Bezout domain for some $c \in R$. Hence M is a maximal ideal which is a principal ideal.

$(2) \Rightarrow (1)$. Suppose that any maximal ideal M that contains an element a, is a principal one. Considering its homomorphic image we have $\overline{M} = \overline{m}\overline{R} = \mathrm{Ann}(\overline{n}\overline{R})$ because R/aR is a morphic ring. Since $\overline{m} \notin U(\overline{R})$ then we have $\mathrm{Ann}(\overline{n}\overline{R}) \neq \overline{R}$ and hence $\overline{n}\overline{R} \neq \{\overline{0}\}$.

As a result $\mathrm{Ann}(\overline{M}) = \mathrm{Ann}(\mathrm{Ann}(\overline{n}\overline{R})) = \overline{n}\overline{R} \neq \{\overline{0}\}$. Therefore $\mathrm{Ann}(\overline{M})$ is a nonzero principal ideal and this proves the fact that R/aR is a Kasch ring. □

In this pioneering paper Kaplansky has raised the question: if $aR = bR$ in a ring R then are a and b necessarily right associates?

Developing these ideas Canfell [8] has introduced the concept of the uniquely generated set of principal ideals.

Definition 5.11. Let $\{a_iR | i = 1,2,\dots,n\}$ be a finite set of principal ideals of commutative ring R. It is said that this set of principal ideals is *uniquely generated* if whenever $a_1R = b_1R, \dots, a_nR = b_nR$ there exist elements $u_1, \dots, u_n \in R$ such that $a_i = b_iu_i$, $i = 1,2,\dots,n$, and $u_1R + u_2R + \dots + u_nR = R$. *The dimension of a commutative ring R* (denoted by $\dim(R)$) is the least integer n such that every set of $n+1$ principal ideals is uniquely generated.

Canfell has obtained the characterizations of n-dimensional F-spaces in terms of the rings of continuous real-valued and complex-valued functions defined on such spaces. Extending the uniqueness notion of principal ideals generators he gave an algebraic characterization of the concept "n-dimensional".

We will show that in the case of a commutative morphic ring the property $\dim(R) = 1$ is equivalent to the stable range 2 condition.

Theorem 5.39. *Let R be a commutative Bezout ring and* $\dim(R) = 1$. *Then* $\text{st.r.}(R) = 2$.

Proof. Let $a, b \in R$. Since R is a commutative Bezout ring, $aR + bR = dR$ for some element $d \in R$. There exist $a_0, b_0 \in R$ and $u, v \in R$ such that $a = da_0$, $b = db_0$ and $d = au + bv = a_0ud + b_0vd$. Taking $q = 1 - a_0u - b_0v$. We see that $dq = 0$ and for any elements $t_1, t_2 \in R$ there are the equalities $(a_0 + t_1q)d = a$, $(b_0 + t_2q)d = b$. We will make a definite choice of t_i, $i = 1, 2$, so that the elements $a_0 + t_1q = a_1$ and $b_0 + t_2q = b_2$ generate R.

Then $a_1x + b_1y = 1$ for some elements $x, y \in R$ and $a = a_1d$, $b = b_1d$. By Theorems **??** R is an Hermite ring and $\text{st.r.}(R) = 2$. Theorem is proved. □

Theorem 5.40. *Let R be a commutative morphic ring of stable range 2. Then* $\dim(R) = 1$.

Proof. Let $a_1R = b_1R$ and $a_2R = b_2R$. Then $a_1 = x_1b_1$, $a_2 = x_2b_2$ and $b_1 = y_1a_1$, $b_2 = y_2a_2$ for some $x_1, x_2, y_1, y_2 \in R$. Then $b_1(1 - x_1y_1) = 0$, $b_2(1 - x_2y_2) = 0$ and $1 - x_1y_1 \in$
$Ann(b_1)$, $1 - x_2y_2 \in$
$Ann(b_2)$. Let $\text{Ann}(b_1) = \alpha_1R$ and $\text{Ann}(b_2) = \alpha_2R$ for some $\alpha_1, \alpha_2 \in R$.

Since $1 - x_1y_1 \in \alpha_1R$ and $1 - x_2y_2 \in \alpha_2R$, we have $x_1R + \alpha_1R = R$ and $x_2R + \alpha_2R = R$. Obviously, $x_1R + x_2R + \alpha_1\alpha_2R = R$. Since $\text{st.r.}(R) = 2$ then $(x_1 + \alpha_1\alpha_2s)R + (x_2 + \alpha_1\alpha_2t)R = R$ for some $s, t \in R$. Since

$$(x_1 + \alpha_1\alpha_2t)b_1 = x_1b_1 + \alpha_2t\alpha_1b = x_1b_1 = a_1,$$

$$(x_2 + \alpha_1\alpha_2s)b_2 = x_2b_2 + \alpha_1s\alpha_2b = x_2b_2 = a_2.$$

Denote $x_1 + \alpha_1\alpha_2t = u_1$, $x_2 + \alpha_1\alpha_2s = u_2$.

We have proved that $u_1b_1 = a_1$, $u_2b_2 = a_2$ and $u_1R + u_2R = R$, that is $\dim(R) = 1$ that theorem is proved. □

As a consequence of Theorems **??** we obtain the following result.

Theorem 5.41. *A commutative morphic ring R is a ring of stable range 2 if and only if* $\dim(R) = 1$.

The notion of principal ideals being uniquely generated first appeared in Kaplansky's classic paper [39]. He had raised the question: when the ideals of the ring R satisfy the property of being uniquely generated. He remarked that for commutative rings the property holds for principal ideal rings and artinian rings. In the case of a left quasi-morphic rings the property of being uniquely generated is equivalent to that a ring has stable range one. We will show that for a commutative morphic ring the condition of a neat range one condition is equivalent to the weak uniquely generation condition by neat elements.

In the following we assume that R is a commutative ring.

Definition 5.12. (a) An element $a \in R$ is *a unit modulo a principal ideal bR* if $ax - 1 \in bR$ for some $x \in R$. (b) A unit $a \in R$ modulo a principal ideal *bR lifts to a neat element*, if $a - t \in bR$ for some neat element $t \in R$.

Proposition 5.13. *Let R be a commutative ring. Then the following are equivalent:*

1. R is a ring of neat range one;

2. every unit lifts to a neat element modulo every principal ideal.

Proof. We assume that R is a ring of neat range one. Let $a, b, c \in R$ be such that $ab - 1 \in cR$, i.e. b is a unit modulo the principal ideal cR. We are going to show that there exists a neat element $t \in R$ such that $b - t \in R$.

Let $x \in R$ be such that $ab - 1 = cx$. Then $ab - cx = 1$. Since R is a ring of neat range one there exists an element $s \in R$ and a neat element $t \in R$ such that $b - cs = t$. Therefore $b - t \in cR$ where t is a neat element in R.

To prove the implication $(2) \Rightarrow (1)$ we assume that every unit of R lifts to some neat element modulo every principal ideal. We are going to show that R is a ring of a neat range one. Let $a, b, c, d \in R$ such that $ab + cd = 1$. Then $ab - 1 \in cR$. By our hypothesis there exists a neat element $t \in R$ such that $b - t \in cR$. Thus $b - t = cx$ for some $x \in R$ i.e. $b + c(-x) = t$ is a neat element, i.e. R has neat range one. □

Proposition 5.14. *A morphic ring is a ring of neat range one if and only if for any pair of elements $a, b \in R$ such that $aR = bR$ there are neat elements $s, t \in R$ such that $as = b$ and $a = bt$.*

Proof. In view of Proposition 5.13 it suffices to show that every unit lifts to a neat element modulo every principal ideal in R.

Let x be a unit that lifts to a neat element modulo the principal ideal yR, i.e there exists $z \in R$ such that $zx - 1 \in yR$. We would like to show that there exists a neat element $t \in R$ such that $x - t \in yR$. Since R is morphic then there exist a, b such that $yR = \text{Ann}(a)$ and $xaR = \text{Ann}(b)$.

Obviously, $xR \subset \text{Ann}(ab)$ and $yR \subseteq \text{Ann}(ab)$.

Since $zx - 1 \in yR$ then we have $xR + yR = R$ and $xR + yR = \text{Ann}(ab)$. Then $ab = 0$ and $a \in \text{Ann}(b)$. Also we have $\text{Ann}(b) = xaR \subseteq aR$. Therefore $\text{Ann}(b) = xaR = aR$. Under the assumption on ring there exists a neat element $t \in R$ such that $xa = ta$. This implies that $(x - t)a = 0$, so $x - t \in \text{Ann}(a) = yR$. Thus Proposition 5.13 implies that R is a ring of neat range one.

Conversely, let $aR = bR$. Then there exist $x, y \in R$ such that $a = bx$, $b = ay$. Therefore $b = bxy$, $b(1 - xy) = 0$. This shows that $1 - xy \in \text{Ann}(b)$.

It is clear that $xy + (1 - xy) = 1$, so $xy \in xR$ and $1 - xy \in (1 - xy)R$. Therefore $xR + (1 - xy)R = R$. Since R is assumed to be of neat range one then there exists $s \in R$ such that $x + (1 - xt)s = t$ is a neat element of R. Since $1 - xy \in \text{Ann}(b)$ then $(x + (1 - xy)s)b = tb$, $xb = tb$, where $xb = a$. Thus $a = tb$ for some neat element $t \in R$. Similarly, $b = sa$ for some neat element $s \in R$ that completes the proof. □

Theorem 5.42. *If R is an elementary divisor domain and $a \in R \setminus \{0\}$, then the quotient-ring R/aR is a morphic ring of neat range one.*

Proof. Since every elementary divisor domain is a Bezout ring then by Theorem 5.12 we know that R/aR is a morphic ring. Since every homomorphic image of an elementary divisor ring is an elementary divisor ring then R/aR is a morphic ring of neat range one as was desired.

Let R be an elementary divisor domain which is not of almost stable range one. Then there exists an element $a \in R$ such that in the quotient-ring $\overline{R} = R/aR$ there exist elements $\overline{b}, \overline{c} \in \overline{R}$ such that $\overline{b}\overline{R} = \overline{c}\overline{R}$. Hence there exist non invertible neat elements $\overline{s}, \overline{t} \in R$ such that $\overline{b}\overline{s} = \overline{c}$, $\overline{c}\overline{t} = \overline{b}$.

5.8 Gelfand range one and Bezout PM*-domains

Recall that in Section **??** we have introduced a notion of Gelfand ring in a commutative case. Here is a definition in a noncommutative one.

Definition 5.13. A *Gelfand ring* is an associative ring R with identity such that if I and J are distinct ideals then there are elements i and j such that $iRj = 0$ i is not in I and j is not J.

Mulvey [46] introduced this notion as rings which one could prove a generalization of Gelfand duality and named them after Israel Gelfand.

In this section we are going to prove that a commutative Bezout domain in which any nonzero prime ideal is contained in a unique maximal ideal is an elementary divisor ring.

Definition 5.14. A nonzero element a of a commutative ring is called *a PM-element* if the quotient-ring R/aR is a PM-ring.

Proposition 5.15. *For a commutative ring R the following statements are equivalent:*

1. *a is a PM-element;*

2. *for each prime ideal P such that $a \in P$ there exists a unique maximal ideal such that $P \subset M$.*

Proof. The equivalence of statements follows from the fact that every prime ideal of $\overline{R} = R/aR$ is of form P/aR, where P is a prime ideal of R such that $a \in P$. □

As a consequence of Proposition 5.15 we obtain the following results.

Proposition 5.16. *A commutative domain R is a domain in which any nonzero prime ideal is contained in a unique maximal ideal if and only if any nonzero element of R is a PM-element.*

Proposition 5.17. *An element a of a commutative Bezout domain R is a PM-element if and only if for every elements $b, c \in R$ such that $aR + bR + cR = R$ an element a can be represented as $a = rs$, where $rR + bR = R$ and $sR + cR = R$.*

Proof. Let $\overline{R} = R/aR$, $\overline{b} = b + aR$, $\overline{c} = c + aR$. Since

$$aR + bR + cR = R$$

then $\overline{b}\overline{R} + \overline{c}\overline{R} = \overline{R}$. Therefore, if $a = rs$, where $rR + bR = R$ and $sR + cR = R$, then $\overline{b}\overline{R} + \overline{c}\overline{R} = \overline{R}$, $\overline{0} = \overline{R}\overline{s}$, $\overline{R}\overline{R} + \overline{b}\overline{R} = \overline{R}$,

$$\overline{s}\overline{R} + \overline{c}\overline{R} = \overline{R},$$

i.e. R is a Gelfand ring. By Theorem 2.3 $\overline{R}$ is a PM-ring. Conversely, let $\overline{R}$ be a PM-ring. This is the same being a Gelfand ring, i.e. $\overline{0} = \overline{R}\overline{s}$ where $\overline{R}\overline{R} + \overline{b}\overline{R} = \overline{R}$, $\overline{s}\overline{R} + \overline{c}\overline{R} = \overline{R}$ for arbitrary $\overline{b}, \overline{c} \in \overline{R}$ such that $\overline{b}\overline{R} + \overline{c}\overline{R} = \overline{R}$.

Whence we obtain $aR+bR+cR=R$. The equality

$$\overline{0}=0+aR=\overline{R}\overline{s}$$

implies that $rs\in aR$, where $\overline{R}=r+aR$, $\overline{s}=s+aR$. Let $rR+aR=r_1R$. The latter yields $r=r_1r_0$, $a=r_1a_0$, where $r_0R+a_0R=R$. Suppose that

$$r_0u+a_0v=1$$

for some $u,v\in R$. Since $rs\in aR$, then $rs=at$ for some $t\in R$. After the substitution of $r=r_1r_0$, $a=r_1a_0$ into the latter equality we obtain

$$r_1r_0s=r_1a_0t,$$

and using the fact that R is a domain we see that $a_0t=r_0s$. From the equality

$$r_0u+a_0v=1$$

we obtain $sr_0u+sa_0v=s$, $a_0(tu+sv)=s$. Therefore $a=r_1a_0$, where

$$r_1R+bR+r_1a_0R=R.$$

Then $r_1R+bR=R$. Since $a_0(tu+sv)=s$ and $a_0R+cR+r_1a_0R=R$ then we obtain $a_0R+cR=R$. Proposition is proved. □

Theorem 5.43. *A commutative Bezout domain in which any nonzero prime ideal is contained in a unique maximal ideal is an elementary divisor ring.*

Proof. Let R be a commutative Bezout domain in which any nonzero prime ideal is contained in a unique maximal ideal. Let $aR+bR+cR=R$. According to the restriction unposed on R by Proposition 5.17 we have that $b=rs$, where $rR+aR=r$, $sR+cR=R$. Let p be such that $sp+ck=1$ for some $k\in R$. Hence $rsp+rck=r$. Denoting $rk=q$ we obtain

$$(bp+cq)R+aR=R.$$

Let $pR+qR=dR$ and $p=dp_1$,$q=dq_1$, where $p_1R+q_1R=R$. Hence

$$p_1R+(p_1b+q_1c)R=R$$

and since $pR\subset p_1R$ we obtain $p_1R+cR=R$, $p_1R+(p_1b+q_1c)R=R$.

Since $bp+cq=d(bp_1+cq_1)$ and $(bp+cq)R+aR=R$ we obtain $(bp_1+cq_1)R+aR=R$. Finally, we get $ap_1R+(bp_1+cq_1)R=R$. By Theorem 3.6 we conclude that R is an elementary divisor ring. Theorem is proved. □

In the case of a commutative Bezout domain this result can be clarified and improved.

Definition 5.15. Let R be a commutative Bezout domain. We say that R is a ring of *Gelfand range 1* if for any $a,b \in R$ such that $aR+bR=R$ there exists an element $t \in R$ such that $a+bt$ is a PM-element.

Since every unit is PM-element we have the following result.

Proposition 5.18. *A commutative Bezout domain of stable range 1 is a ring of Gelfand range 1.*

Since every avoidable element is a Gelfand element by Proposition 8.1 and Theorem 5.21 we obtain the theorems:

Theorem 5.44. *Elementary divisor domain R is a ring of Gelfand range 1.*

Theorem 5.45. *Let R be a commutative Bezout domain of Gelfand range 1. Then R is an elementary divisor ring.*

Proof. It is sufficient to diagonalize a matrix

$$A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$$

where $aR+bR+cR=R$ by Theorem 3.4.

Write $ax+by+cz=1$ for some elements $x,y,z \in R$. Then $bR+(ax+cz)R=R$. Since R is a ring of Gelfand range 1 there exists some $t \in R$ such that $d=b+(ax+c)t$ is a PM-element.

Using elementary transformations we obtain a PM-element d in a matrix

$$\begin{pmatrix} 1 & 0 \\ xt & 1 \end{pmatrix} A \begin{pmatrix} 1 & 0 \\ zt & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ d & c \end{pmatrix},$$

where $aR+dR+cR=R$ and d is a PM-element.

According to the restrictions on d there is a decomposition $d=rs$, where $rR+aR=R$, $sR+cR=R$. Let $p,k \in R$ be such that $sp+ck=1$ for some element $k \in R$. Hence $rsp+rck=r$ and $dp+crk=c$. Denoting $rk=q$ we obtain $(dp+cq)R+aR=R$. If $pR+qR=\delta R$ then $p=p_1\delta$, $q=q_1\delta$ and $p_1R+q_1R=R$ for some $p_1,q_1 \in R$.

Since $p_1R+cR=R$ and $p_1R+q_1R=R$ then

$$p_1R+(p_1d+q_1c)R=R.$$

Moreover, the equalities $dp+cq = \delta(dp_1+cq_1)$ and $(dp+cq)R+aR = R$ imply that $(dp_1+cq_1)R+aR=R$. Reminding that $(dp_1+cq_1)R+p_1R=R$, finally we obtain $(dp_1+cq_1)R+ap_1R=R$. Since p_1 and q_1 are coprime then a column

$$\begin{pmatrix} p_1 \\ q_1 \end{pmatrix}$$

can be completed to some invertible matrix P. Hence

We have

$$\begin{pmatrix} a & 0 \\ d & c \end{pmatrix} P = \begin{pmatrix} ap_1 & * \\ dp_1+cq_1 & * \end{pmatrix} = B.$$

Obviously a matrix B (and hence matrix A) admits canonical diagonal reduction, i.e. R is an elementary divisor ring. Theorem is proved. □

5.9 Lattice-ordered groups and Montgomery counterexample

Larsen M. D., Lewis W. J., Shores T. S. [43] asked the following question: if R is a Bezout domain with the property that every nonzero prime ideal is contained in a unique maximal ideal then is R necessarily an adequate ring?

In our approach we use the group of divisibility of a commutative Bezout domain.

Definition 5.16. For a domain R we denote by K its classical field of fractions and by K^* the set of nonzero elements of K. K^* is an abelian group under multiplication and $U(R)$ is its subgroup. We define

$$G(R) = K^*/U(R)$$

and call it as *group of divisibility* of R. On the set $G(R)$ we can introduce the partial order by the relation

$$aU(R) \leq bU(R) \Leftrightarrow b|a \in R \in G(R)$$

for any $aU(R) \leq bU(R) \in G(R)$. This definition is well-defined and makes $G(R)$ into a partially-ordered group. The *positive cone* is the set of elements

$$aU(R) \in G(R)$$

such that

$$1U(R) \leq aU(R),$$

i. e. it is the set of cosets whose representative belong to R. We denote the positive cone of a partially ordered group G by G^+.

The partial-order defined above becomes a lattice-order and makes $G(R)$ into a *lattice-ordered group* (or *l-group* for short) precisely, when R is a Bezout domain.

The following well-known theorem states that every abelian l-group may be realized as the group of divisibility of some Bezout domain. For some nice references about lattice-ordered groups see **[ ]**.

Theorem 5.46. ***(Jaffard-Ohm-Kaplansky)*** *Let G be an abelian l-group. There exists a Bezout domain R such that $G(R) \cong G$.*

In this group the operation will be written additively. Let G be a lattice-ordered group. For notational convenience, let

$$G^+ = \{x \in R | x \geq 0\}.$$

Definition 5.17. We say that G is *adequate* if for every $a, b \in G^+$ if there exist $r, s \in G^+$ such that $a = r + s$, $r \wedge b = 0$ and if $0 < s' \leq s$ for some $s' \in G$ then $s' \wedge b \neq 0$.

It is easy see that we have simply translated the Bezout ring adequacy condition into the language of ordered groups. We immediately have the fact that a Bezout domain is adequate if and only if its group of divisibility is adequate.

Let G be a lattice-ordered group and let $b \in G^+$. We define

$$G_b{}^+ = \{a \in G^+ | a \wedge b = 0\}, \;\; G_b = \{a_1 - a_2 | a_1, a_2 \in G_b{}^+\}.$$

Note that if

$$a_1 \wedge b = 0, \;\; a_2 \wedge b = 0, \quad \Rightarrow (a_1 + a_2) \wedge b = 0,$$

so that G_b is a lattice subgroup of G.

Definition 5.18. It is said that an l-group is *projectable* if for each $b \in G^+$, G_b is a summand of G (see Theorem 5.47).

Theorem 5.47. *Let G be a lattice ordered group. Then G is adequate if and only if for each $b \in G^+$, G_b is a summand of G.*

Proof. Let G be adequate and take any $b \in G^+$. Let

$$H_b{}^+ = \{a \in G^+ | a' \wedge b \neq 0 \text{ whenever } 0 < a' \leq a\}.$$

If $a_1, a_2 \in H_b{}^+$ and $0 < s \leq a_1 + a_2$ for some $s \in G$, then either $s \wedge a_1 \neq 0$ or $s \wedge a_2 \neq 0$; hence $s \wedge b \neq 0$ and $a_1 + a_2 \in H_b{}^+$. Thus

$$H_b = \{a_1 - a_2 | a_1, a_2 \in H_b{}^+\}$$

is a lattice-ordered subgroup of G. Clearly $G_b \cap H_b = \{0\}$.

If $a \in G^+$ then $a = r + s$ for some $r \in G_b$ and $s \in H_b$ since G is adequate. Since any element of a lattice ordered group is the difference of two positive elements, we get that

$$G = G_b \oplus H_b.$$

Conversely suppose that $G = G_b \oplus H$ for some lattice subgroup H of G.

Let $a \in D^+$ and write $a = r + s$ for some $r \in G_b{}^+$, $s \in H^+$. Then $r \wedge b = 0$. Suppose $0 < s' \leq s$ for some $s' \in G$. Then $s' \in H$ and hence $s' \wedge b \neq 0$. Thus G is adequate. Theorem is proved. □

It is well known **[ ]** that any lattice-ordered group can be lattice-embedded into a product of totally ordered groups in such way that infimums are preserved. Both a direct product and a direct sum of totally ordered groups are adequate groups, as we can easily prove the following result.

Proposition 5.19. *Let R be a Bezout domain. Then R is adequate if its group of divisibility is order-isomorphic to either a direct sum or direct product of totally ordered group.*

By Theorem 5.46 if R is a Bezout domain, then its divisibility group $G(R)$ is an l-group and if G is any abelian l-group, there exists a Bezout domain R such that $G(R) \cong G$. Moreover, there is an poset anti-isomorphism between the set of prime ideals of a Bezout domain and the set of prime subgroups of its divisibility group.

Definition 5.19. A subgroup S of an l-group G in an *l-subgroup* provided that S is a sublattice of G, and S is a *convex* l-subgroup if $0 < g, s \in S$ and $g \in G$ imply that $g \in S$.

Definition 5.20. A convex l-subgroup S of an l-group G is *prime* if G/S is totally ordered or equivalently if $a \wedge b = 0$ in G then $a \in S$ or $b \in S$.

Example 5.2. (**Montgomery's example**) [5] In this example we will construct a Bezout domain R such that any its prime ideal is contained in a unique maximal ideal, but R is not adequate. Our goal is to find an abelian l-group G that satisfies:

1. Each proper prime subgroup of G contains unique minimal prime subgroup;

2. G is not projectable.

Consider the following partially ordered set

$$\Gamma : \quad \begin{array}{ccccc} 0 & \bullet & \longrightarrow & \bullet & 1 \\ 1 & \bullet & \longrightarrow & \bullet & 2 \\ 2 & \bullet & \longrightarrow & \bullet & 3 \\ 3 & \bullet & \longrightarrow & \bullet & 4 \\ 4 & \bullet & \longrightarrow & \bullet & 5 \\ & \bullet & \longrightarrow & \bullet & \\ & \bullet & \longrightarrow & \bullet & \\ & \bullet & \longrightarrow & \bullet & \end{array}$$

Let V be the set of all integer valued functions on Γ and define

$$v = (v_0, v_1, \ldots)$$

to be positive if $v_0 > 0$ and $v_i \geq 0$ where $i = 2, 3, \ldots$ or $v_0 = 0$ and $v_i \geq 0$ for $i = 1, 2, 3, \ldots$. Then V is an l-group. Let G be the subgroup of V generated by the small or restricted direct sum on $1, 2, 3, \ldots$ and the element $a = (1, 0, 0, 1, 0, \ldots)$. It is easy to checked that if $g \in G$ then $g \vee 0 \in G$ and hence G is an l-subgroup of V.

Let $V(i)$ be the characteristic function on i for $i = 1, 2, 3, \ldots$. Then

$$G = G_{v(i)} \oplus H_{v(i)}$$

for $i = 2, 3, \ldots$, but $a \in G \setminus G_{v(i)} \oplus H_{v(i)}$, so that G does not projectable.

Let $U_0 = \{v \in G | v_0 = 0\}$, $U_1 = \{v \in G | v_0 = v_1 = 0\}$, $U_i = \{v \in G | v_i = 0\}$ for $i = 2, 3, \ldots$.

It can be easily checked that G/U is totally ordered, so that each U_i is a proper prime subgroup of G. Suppose that P is a prime subgroup of G. If $v(i) \in P$ for $i = 2, 3, \ldots$ then $P \supseteq U_1$ and hence $P = U_0$ or $P = U_1$. If $v(i) \notin P$ for some $i > 1$, then $U_i \subseteq P$ and hence $P = U_i$. Thus $\{U_i\}_{i=0}^{\infty}$ is the set of all proper prime subgroups of G and hence G is an l-group such that each proper prime subgroup contains a unique minimal prime subgroup.

Let R be a Bezout domain whose group of divisibility is isomorphic to constructed l-group G. Then R is a Bezout domain each nonzero prime ideal contained in a unique maximal ideal and R is not adequate. □

Definition 5.21. Let G be an l-group (written multiplicatively).

1) We say $u \in G^+$ is a *(weak) order unit* if whenever $u \wedge x = 1$ then it follows that $x = 1$. It is known that a positive element of G is an order unit precisely when it does not belong to any minimal prime subgroup of G.

2) Let $x \in G^+$. If there exists an element y such that $x \wedge y = 1$ and $x \vee y$ is an order unit then x is said to be a *complemented element* of G.

3) If every positive element of G is complemented, then G is said to be a *complemented l-group*.

4) G is called *locally complemented* if for each $g \in G^+$ the convex l-subgroup generated by g (denoted by $G(g)$) is complemented.

5) If every prime subgroup of G contains unique minimal subgroup then it is said that G has *stranded primes*.

Proposition 5.20. [ ] *Suppose that R is a commutative Bezout domain. R is a neat ring if and only if $G(R)$ is locally weakly complemented and has stranded primes.*

Proposition 5.21. [ ] *A commutative Bezout domain R satisfies Henriksen hypothesis if and only if $G(R)$ is a locally complemented l-group.*

5.10 Rings of continuous functions C(X)

Recall some basic information from topology that will be used for our purposes in the following.

Definition 5.22. Let X be a topological space. We say that a subset W of X is *clopen* if it is both closed and open in X. A topological space is called *zero-dimensional* if it has a base of clopen sets, and *strongly zero-dimensional* if any two disjoint closed sets are contained in some disjoint clopen sets. Finally, we say that X is *basically disconnected* if any two disjoint open sets have disjoint closures.

In the following by $C(X)$ we mean the ring of all real-valued continuous functions on topological space X.

Definition 5.23. Suppose X is a topological space. A subset $Z \subseteq X$ is called a*zero-set* of X if there is $f \in C(X)$ such that

$$Z = \{x \in X | f(x) = 0\}.$$

A *cozero-set* is the (set theoretic) complement of a zero-set.

Note that $C(X)$ is an abelian l-group under the point-wise operations and the constant function 1 is a weak-order unit **[ ]**.

Definition 5.24. The topological space X such that every l-subgroup of $C(X)$ contains a unique minimal prime subgroup is called F*-spaces*

In the following we assume that our topological spaces are Tychonoff (completely regular and Hausdorff).

Recently we have shown that for any element a of an adequate domain R the quotient-ring R/aR is a clean ring then every adequate domain is a neat one. The converse isn't true as we will see in the following example.

Example 5.3. **[ ]** Let $G = C(\beta\mathbb{N} \setminus \mathbb{N})$, where $\beta\mathbb{N}$ denotes the Stone-Čech compactification of the natural numbers. Since the space $\beta\mathbb{N} \setminus \mathbb{N}$ is a strongly zero-dimensionall F-space and is not basically disconected then it follows that G is weakly complemented and has stranded primes but is not complemented **[ ]**. Therefore, if R is a commutative Bezout domain whose group of divisibility is isomorphic to G then R is a neat Bezout domain an hence an avoidable ring. Since G is not complemented R does not satisfy Henriksen hypothesis an hence R is not an adequate ring.