Chapter 6
Bezout domains and their overrings

6.1 Almost stable range one

In this section we assume that R be a commutative ring.

Following McGovern a ring R has *almost stable range 1* provided that every proper homomorphic image of R has stable range 1. We consider the concept of element of almost stable range 1 which will play the central role in the following studies.

Definition 6.1. An element a called *an element of stable range 1* if for any element $b \in R$ such that $aR + bR = R$ there exists $t \in R$ such that $(a + bt)R = R$.

Proposition 6.1. *Let R be a commutative ring. Then any idempotent is an element of stable range 1.*

Proof. Let $e^2 = e$ and $eR + bR = R$. Then $eu + bv = 1$ for some elements $u, v \in R$. Hence

$$(1 - e)eu + (1 - e)bv = 1 - e.$$

Since e is an idempotent then

$$e + b(1 - e)v = 1$$

i.e. e is an element of stable range 1. Proposition is proved. □

Proposition 6.2. *Let R be a commutative Bezout ring. Then a set of all elements of stable range 1 is a multiplicatively closed set.*

Proof. Let a, c be the elements of stable range 1. Prove that for any element $b \in R$ such that $acR + bR = R$ there exist element $t \in R$ such that $(ac + bt)R = R$.

Since $acR + bR = R$ then $aR + bR = R$ and $cR + bR = R$. As a and c are the elements of stable range 1 then

$$a+bx=u_1, \quad c+by=u_2$$

for some $x,y\in R$ where u_1,u_2 are some units of a ring R. Then

$$u_1u_2=(a+bx)(c+by)=ac+b(ay+cx+bxy).$$

Since u_1u_2 is unit in R then we obtain that ac is an element of stable range 1. Proposition is proved. □

Definition 6.2. An element a of a ring R is called *an element of almost stable range 1* if the quotient-ring R/aR is a ring of stable range 1.

Obviously any element of stable range 1 is an element of almost stable range 1.

Moreover, if R is a commutative principal ideal domain (for example ring of integers), which is not of stable range 1, then every nonzero element of R is an element of almost stable range 1. We have more general result.

Proposition 6.3. *Let R be a commutative ring of Krull dimension 1. Then every non divisor zero element (we call such elements regular) is an element of almost stable range 1.*

Proof. Let a be a regular element in a ring R. Then element a does not contain in any minimal prime ideal. And it means that a factor-ring R/aR is a ring of Krull dimension of zero. Then

$$(R/aR)/(J(aR)/aR)\cong R/J(aR)$$

is a von Neumann regular ring, where $J(aR)$ – Jacobson radical of aR.

Since the stable range of a commutative von Neumann regular ring equals 1 and

$$\text{st.r.}(R/aR)=\text{st.r.}(R/J(aR))$$

then any element $a\in R$ is an element of almost stable range 1. Proposition is proved. □

Proposition 6.4. *Let a be an element of almost stable range 1 in a commutative ring R. If $aR+bR+cR=R$ then there exists an element $y\in R$ such that $aR+(b+cy)R=R$.*

Proof. Let $\overline{R}=R/aR$ and $aR+bR+cR=R$. Let $\overline{x}=x+aR$ for some $x\in R$. Since

$$\overline{b}\overline{R}+\overline{c}\overline{R}=\overline{R}$$

there is $\overline{y} \in \overline{R}$ such that $(\overline{b}+\overline{cy})\overline{R} = \overline{R}$. Let's show that

$$aR + (b + cy)R = R,$$

where $\overline{y} = y + aR$. Let M be any maximal ideal of R such that

$$aR + (b + cy)R \subset M.$$

But this is impossible, since $\overline{M} = M/aR$ is a maximal ideal of $\overline{R}$ such that $\overline{b+cy} \in \overline{M}$. Therefore $aR + (b + cy)R = R$. Proposition is proved. □

Proposition 6.5. *Let a be an element of R such that for any $b, c \in R$ such that $aR + bR + cR = R$ there exists an element $y \in R$ such that $aR + (b + cy)R = R$. Then a is an element of almost stable range 1.*

Proof. Let $\overline{R} = R/aR$ and $\overline{b}\overline{R} + \overline{c}\overline{R} = \overline{R}$ where $\overline{b} = b + aR$, $\overline{c} = c + aR$. Obviously $aR + bR + cR = R$. Since $\overline{b}$ and $\overline{c}$ are coprime then

$$(\overline{b}+\overline{cy})\overline{R} = \overline{R},$$

i.e. $sr(R/aR) = 1$ that was desired. □

Definition 6.3. A commutative ring in which every nonzero element is an element of almost stable range 1 is called a ring of *almost stable range 1*.

The first and the simplest example of almost stable range 1 ring is a ring of stable range 1. Also, every commutative principal ideal ring that is not a ring of stable range 1 (for example the ring of integers) is a ring of almost stable range 1 which is not a ring of stable range 1.

Theorem 6.1. *Let R be a ring of almost stable range 1. If $J(R) \neq 0$ then R is a ring of stable range 1.*

Proof. Let $b, c \in R$ be such that $bR + cR = R$ and $a \in J(R) \setminus \{0\}$. Then $aR + bR + cR = R$. By Proposition 6.4 there exists an element $y \in R$ such that $aR + (b + cy)R = R$. Since $a \in J(R)$, then $(b + cy)R = R$, i.e. st.r.$(R) = 1$. Theorem is proved. □

Theorem 6.2. *Let R be a commutative Bezout domain and a is an avoidable element. Then a is an element of almost stable range 1.*

Proof. By Proposition 5.2 and Theorem 5.24 the ring $\overline{R} = R/aR$ is clean and by Proposition 2.17 it is a ring of idempotent stable range 1. Hence $a \in R$ is an element of an almost stable range 1 as was desired. □

Theorem 6.3. *Any nonzero element of arbitrary adequate ring is an element of almost stable range 1.*

Proof. Since any adequate element is an avoidable element then the result follows from Theorem 6.2. □

Definition 6.4. A commutative ring R is a ring of *quasi-stable range 1* if any element $a \in R \setminus J(R)$ is an element of almost stable range 1.

Obviously any ring of almost stable range 1 is a ring of quasi-stable range 1.

Example 6.1. (**Henriksen's example**) Recall the ring in Example 4.3

$$R = \{z_0 + a_1 x + a_2 x^2 + \dots | z_0 \in \mathbb{Z}, a_i \in \mathbb{Q}\}$$

This ring is a ring of quasi-stable range 1 but it is not a ring of almost stable range 1.

By Theorems 3.2,3.6 and 5.25 we have the following results.

Theorem 6.4. *An avoidable ring is an elementary divisor ring if and only if it is a Bezout ring.*

Theorem 6.5. *A ring of quasi stable range 1 is an elementary divisor ring if and only if it is a Bezout ring.*

Theorem 6.6. *A commutative ring of almost stable range 1 is an elementary divisor ring if and only if it is a commutative Bezout ring.*

Let R be a commutative ring. Consider a set $S = \{a \in R \mid \text{st.r.}(R/aR) = 1\}$. Obviously $S \neq \emptyset$ since $1 \in S$.

Proposition 6.6. *Set S is a saturated multiplicatively closed set.*

Proof. Let $a, b \in S$, i.e st.r.$(R/aR) = 1$ and st.r.$(R/bR) = 1$. By **[ ]** st.r.$(R/(aR \cap bR)) = 1$. Since

$$R/(aR \cap bR) \cong R/(abR)/(aR \cap bR)/(ab)R$$

then the quotient-ring at the right hand side is a ring of stable range one:

$$\text{st.r.}(R/(ab)R/(aR \cap bR)/(ab)R) = 1.$$

Since $(aR \cap bR)/abR$ is a nil ideal then

$$(aR \cap bR)/(ab)R \subset J(R/(ab)R).$$

As the stable range of ring is preserved in the quotients by nil ideals then st.r.$(R/(ab)R) = 1$. Hence $ab \in S$.

Assume $a = bc$ with $a \in S$. Since $aR \subset bR$ we see that

$$R/bR \cong (R/aR)/(bR/aR)$$

whence st.r.$(R/bR) = 1$. Thus $b \in S$. Proposition is proved. □

Let R be a commutative Bezout domain. For any S is a saturated multiplicatively closed sets we can consider a localization of R with respect to the set S, i.e. the ring of fractions R_S

Theorem 6.7. *Let S be a set of elements a of commutative Bezout domain R such that* st.r.$(R/aR) = 1$. *Then R is an elementary divisor ring if and only if R_S is an elementary divisor ring.*

Proof. By Theorem 3.3 it sufficiently to provide the canonical diagonal form of matrix

$$A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix},$$

where $aR + bR + cR = R$. If R_S is an elementary divisor ring. Then, for elements $a, b, c \in R_S \cap R$ we can find elements $ps^{-1}, qs^{-1} \in R_S$, $s \in S$ such that

$$aps^{-1}R_S + (bps^{-1} + cqs^{-1})R_S = R_S.$$

Using the definition of localization with respect to S we can find some $r, t \in R$ such that

$$(apr + (bp + cq)t) \in S.$$

Since S is a saturated set then we have that A is equivalent to some $B = \begin{pmatrix} z & 0 \\ x & y \end{pmatrix}$ where $xR + yR + zR = R$ and $z \in S$.

By Proposition 6.4, since $z \in S$, we obtain the equality

$$zR + (x + y\lambda)R = R$$

for some $\lambda \in R$. By Theorem 3.6 B (and hence A) admits the canonical diagonal reduction. Therefore R is an elementary divisor ring.

Conversely, assume that R is an elementary divisor ring. It is necessary to show that R_S is also an elementary divisor ring. Consider arbitrary elements

$as^{-1}, bs^{-1}, cs^{-1} \in R_S$ such that $as^{-1}R_S + bs^{-1}R_S + cs^{-1}R_S = R_S$. Then $aR + bR + cR = dR$ for some element $d \in S$. Let $a = a_1 d$, $b = b_1 d$, $c = c_1 d$ for some elements $a_1, b_1, c_1 \in R$ such that $a_1 R + b_1 R + c_1 R = R$ as R is an Hermite ring. By Theorem **??** Since R is an elementary divisor ring, then there exist elements $p, q \in R$ such that

$$a_1 pR + (b_1 p + c_1 q)R = R.$$

Multiplying the latter by s^{-1} we obtain

$$a_1 ps^{-1}R_S + (b_1 ps^{-1} + c_1 qs^{-1})R_S = R_S.$$

By Theorem 3.6 R_S is an elementary divisor ring. Theorem is proved. □

In the remaining part of this section we are going to construct a so called D-chain of saturated multiplicatively closed sets in order to determine the influence of almost stable range 1 elements of being R an elementary divisor ring.

Since S is a saturated multiplicatively closed set apply the transfinite induction and construct a chain we can

$$\{R_\alpha | \alpha \text{ is ordinal}\}$$

of saturated multiplicatively closed sets in the domain R in the following way. Denote set $R^0 = S$. Let α be an ordinal greater than zero and assume that R^β are already constructed saturated multiplicatively closed sets in R for $\beta < \alpha$ and $K_\beta = R_{R^\beta}$. Then K_β is a Bezout domain and $S(K_\beta)$ is a set of all elements of almost stable range 1 of K_β. By Proposition 6.6 $S(K_\beta)$ is saturated and multiplicatively closed. We define

$$R^\alpha = \begin{cases} \bigcup\limits_{\beta<\alpha} R^\beta, & \alpha \text{ is a limit ordinal ;} \\ S(K_{\alpha-1}) \cap R, & \text{otherwise.} \end{cases}$$

Clearly by R^α is again a saturated multiplicatively closed set and if α and β are ordinals such that $\alpha \leq \beta$, then

$$R^\alpha \subset R^\beta \subset R.$$

In addition $R^\alpha = R^{(\alpha+1)}$ for some ordinal α. In fact in the case when

$$R^\alpha \neq R^{(\alpha+1)}$$

for each ordinal α we have

$$\text{card}(R^\alpha) > \text{card}(\alpha).$$

If we choose β such that $\text{card}(\beta) > \text{card}(R)$, then we obtain

$$\text{card}(\beta) > \text{card}(R) > \text{card}(R^\beta)$$

which is a contradiction. Now let α_0 be the least ordinal with the property

$$R^{\alpha_0} = R^{\alpha_0+1}.$$

We say that

$$\{R^\alpha | 0 \leq \alpha \leq \alpha_0\}$$

is a *D-chain* in R.

Let R be a commutative Bezout domain and let S be the set of all its elements of almost stable range 1.

Analyzing the latter theorem and the construction of D-chain in a commutative Bezout domain, we explain why for a commutative Bezout domain being an elementary divisor ring is equivalent to the case of domain with trivial (only units) elements of almost stable range 1.

Let R is a commutative Bezout domain in which every element of almost stable range 1 is a unit, i.e $U(R) = S$. What conditions R has to satisfy in order to be an elementary divisor ring?

By Proposition 8.1 R is an elementary divisor ring if and only if R is a ring of neat range 1, i.e. $aR + bR = R$ implies the existence element $\lambda \in R$ such that $a + b\lambda$ is an avoidable element. Every avoidable element is an element of almost stable range 1. Since $U(R) = S$, then $a + b\lambda$ is a unit, i.e. R is a ring of stable range 1. Since any element of stable range 1 is element of almost stable range 1 we obtain the following result.

Theorem 6.8. *Let R be a commutative Bezout domain such that $U(R) = S$. Then R is an elementary divisor ring if and only if R is a field.*

Hence the problem "Is every a commutative Bezout domain elementary divisor ring" can be reformulated as follows:

Does every commutative Bezout domain contain a non unit element of almost stable range 1?

Is it true that every commutative Bezout domain contains a nonunit element of almost stable range 1?

6.2 Finite localizing embeddings

Let R be a commutative Bezout ring and $a \in R$ be a nonzero and nonunit element of R. We denote by S_a the set

$$S_a = \{b \in R \mid bR + aR = R\}.$$

It is nonempty as always $1R + aR = R$, i.e. $1 \in S_a$.

Proposition 6.7. *Let R be a commutative Bezout domain. Then S_a is a saturated multiplicatively closed set.*

Proof. Let $bR + aR = R$ and $cR + aR = R$ then $bu_1 + av_1 = 1$, $cu_2 + av_1 = 1$ for some elements $u_1, u_2, v_1, v_2 \in R$. If we multiply together these equalities then we obtain

$$bcu_1u_1 + a(bu_1u_2 + cu_2v_1 + av_1v_2) = 1$$

that is the same as $bcR + aR = R$. In the case when $bR + aR = R$ and $b = cd$ for some elements $c, d \in R$ then we obtain $bu + av = 1$ for some elements $u, v \in R$, and $c(du) + av = 1$, as well as $d(cu) + av = 1$. In other words $cR + aR = R$, $dR + aR = R$. Proposition is proved. □

For an arbitrary nonzero and nonunit element of a commutative Bezout domain we denote by R_a a localization of R with the denominators from set S_a.

Proposition 6.8. **[ ]** *Let R be a commutative Bezout domain and let a be a nonzero and nonunit element of R. Then the following statement are equivalent:*

1. $\text{st.r.}(R_a) = 1$;

2. $\text{st.r.}(R/aR) = 1$.

Proof. Let $\overline{R} = R/aR$ and $\overline{b}\overline{R} + \overline{c}\overline{R} = \overline{R}$ for some $\overline{b} = b + aR$, $\overline{c} = c + aR$. Then $aR + bR + cR = R$. Since R is a commutative Bezout domain then $bR + cR = dR$ for some element $d \in R$, i.e. there exist some elements $u, v \in R$ such that $au + bv = d$.

As $aR + bR + cR = R$ we can write $aR + dR = R$, i.e., $aR + (bu + cv)R = R$. This implies that $(bu + cv) \in S_a$ and

$$\frac{b}{1}R_a + \frac{c}{1}R_a = R_a.$$

Since $\text{st.r.}(R_a) = 1$ then one can find $\frac{x}{y} \in R_a$ such that $\frac{b}{1} + \frac{c}{1}\frac{x}{y} = u$ is a unit element of R_a (note that $y \in S_a$). Then $by + cx \in S_a$. Hence $(by + cx)R + aR = R$. Since

$y \in S_a$ we have that $\overline{y} = y + aR$ is a unit element of $\overline{R}$. This means that $\overline{b} + \overline{cxy}^{-1}$ is a unit element of $\overline{R}$. Therefore st.r.$(\overline{R}) = 1$.

To prove the converse statement assume that

$$\frac{b}{s_1}R_a + \frac{c}{s_2}R_a = R_a$$

for some $b, c \in R$ and $s_1, s_2 \in S_a$. Then there exists some $s \in S$ and $u, v \in R$ such that $bu + cv = ss_1s_2 \in S_a$. Hence $aR + (bu + cv)R = R$ and so $\overline{bR} + \overline{cR} = \overline{R}$. Since st.r.$(\overline{R}) = 1$ there is $\overline{y} \in \overline{R}$ such that $\overline{b} + \overline{cy}$ is a unit element of $\overline{R}$. This gives us $(b + cy)R + aR = R$. Hence

$$\frac{b}{s_1}R_a + \frac{c}{s_2}R_a = (\frac{b + cy}{s_3})R_a$$

for some $s_3 \in S_a$. Therefore st.r.$(R_a) = 1$. Proposition is proved. □

Therefore, by Theorem **[ ]** we obtain the theorems.

Theorem 6.9. *Let R be an adequate Bezout domain. Then* st.r.$(R_a) = 1$ *for any nonzero and nonunit element $a \in R$.*

Theorem 6.10. *A commutative Bezout domain is an elementary divisor ring if and only if for any nonzero elements $a, b \in R$ such that $aR + bR = R$ there exist an element $t \in R$ such that satisfies the equality* st.r.$(R_{a+bt}) = 1$.

Proposition 6.9. *Let R be a commutative Bezout domain and a be a nonzero and nonunit element of R. Then $a \in J(R_a)$.*

Proof. For any $\frac{b}{s} \in R_a$ one can write

$$1 - \frac{b}{s}\frac{a}{1} = \frac{s - ab}{s}.$$

Since $s \in S_a$ we have $sR + aR = R$ then $su + av = 1$ for some element $u, v \in R$ and so

$$(s - ab)u + a(v + bu) = 1,$$

i. e., $(s - ab)R + aR = R$. Hence $1 - \frac{b}{s}\frac{a}{1}$ is unit element of R_a. Hence $a \in J(R_a)$. Proposition is proved. □

Proposition 6.10. *Let R be a commutative Bezout domain and a is its nonzero and nonunit element such that and $J(R_a)$ is a principal ideal. Then* st.r.$(R_a) = 1$

Proof. Inasmuch R_a is a commutative Bezout domain and $J(R_a)$ is principal, then $R_a/J(R_a)$ is coherent ring **[ ]**. Since $R/J(R_a)$ is a reduced ring then it is zero-dimensional. The latter means that st.r.$(R_a/J(R_a)) = 1$ and therefore st.r.$(R_a) = 1$. Proposition is proved. □

In this section we will prove that if R is a commutative adequate domain then R is the domain of stable range 1 in localization in multiplicative closed set which corresponds s-torsion in the sense of Komarnitskii.

A question of quasi-reduction of matrices over a commutative domain with so-called L_φ condition is considered by J. Szucs [64]. B. Zabavsky [82] proved that L_φ condition for commutative Bezout domain is nothing else than the stable range condition. Moreover, it is shown how increase the possibilities of reduction in the localization of given ring.

Speaking precisely in this work we are going to prove that an adequate domain is the Bezout domain of stable range 1 in localization by multiplicatively closed set which corresponds to s-torsion in the sense of Komarnitskii [??????].

J. S. Golan, *On S-torsion in the sens of Komarnitskii*, Univer. Haifa, Israll, 1984, prerpint.

Definition 6.5. Let R be a commutative Bezout domain and a is its nonzero and non invertible element. We denote by

$$S_a = \{b \mid b \in R, aR + bR = R\}$$

the set of all elements that are coprime to a and by

$$R_a = RS_a^{-1}$$

the localization of R with respect to this set.

Proposition 6.11. *The set S_a is saturated and multiplicative closed.*

Proof. Let $c, b \in S_a$. By the definition of the set S_a there exist elements $u_1, u_2, v_1, v_2 \in R$ such that

$$au_1 + bv_1 = 1,$$
$$au_2 + bv_2 = 1.$$

Multiplying together these equalities we obtain

$$aw_1 + cbw_2 = 1$$

for some elements $w_1, w_2 \in R$. Therefore $cb \in S_a$.

If $b = cd \in S_a$, then

$$au + c(dv) = 1$$

for some elements $u, v \in R$. So $c \in S_a$ and we conclude that S_a is saturated and multiplicative closed.

Proposition 6.12. *If a is an adequate element of domain R then $st.r.(R_a) = 1$.*

Proof. Let

$$\frac{b}{s}R_a + \frac{c}{s}R_a = R_a.$$

Then

$$\frac{b}{s} \cdot \frac{u}{s_1} + \frac{c}{s} \cdot \frac{v}{s_2} = t,$$

where $s_1, s_2, t \in S_a$. Hence $bu' + cv' = ss_1s_2t \in S_a$ for some $u', v' \in R$. So $(bu + cv)R + aR = R$ and therefore $aR + bR + cR = R$.

Since element a is adequate then by the proof of Theorem 4.4 there exists an element $r \in R$ such as $aR + (b + cr)R = R$. The latter means that $u = b + cr \in S_a$ or equivalently $(b + cr)R_a = R_a$. Moreover,

$$\frac{b}{s} + \frac{c}{s} \cdot \frac{r}{1} = su \in R_a,$$

that is

$$(\frac{b}{s} + \frac{c}{s} \cdot \frac{r}{1})R_a = R_a$$

hence the stable range of the ring R_a equals 1.

Proposition 6.12 immediately implies:

Proposition 6.13. *Let R be an adequate domain. Then for any nonzero and non invertible element $a \in R$ the set R_a is a commutative Bezout domain of the stable range 1.*

Naturally arises the converse question, i.e. to describe precisely the commutative Bezout domains such that each R_a is a commutative Bezout domain of stable range 1.

It is worth to remind that by Theorem 3.2 the stable range of commutative Bezout domain cannot exceed 2, so st.r.$(R_a) \leq 2$ for any nonzero and non invertible element $a \in R$.

Proposition 6.14. *Let R be a commutative Bezout domain such that for any nonzero and non invertible element $a \in R$* $\mathrm{st.r.}(R_a) = 1$. *Then for any nonzero coprime elements $a, b \in R$ there exist $p, q \in R$ such that*

$$aR + bR = (ap + bq)R, \quad pR + (ap + bq)R = R.$$

Proof. Let for any nonzero and non invertible element $x \in R$ the stable range of R_x equals 1 and a, b are not coprime elements of $R \setminus \{0\}$. Since R is a Bezout domain then there exists not invertible $d \in R$ such that $dR = aR + bR$. Furthermore, there are $u, v, a_0, b_0 \in R$ such that $au + bv = d$, $a = da_0$, $b = db_0$.

Since R and R_d are domains then

$$a_0u + b_0v = 1 \Rightarrow a_0R + b_0R = R \Rightarrow a_0R_d + b_0R_d = R_d.$$

Since d is not invertible then $\mathrm{st.r.}(R_d) = 1$ and there are elements $q \in R$ and $u, p \in S_d$ such that

$$\frac{a_0}{1}\frac{q}{p} + \frac{b_0}{1} = u.$$

Hence $a_0q + b_0p = up \in S_a$ according to Proposition 6.11. Therefore

$$(a_0p + b_0q)R + dR = R, \quad pR + dR = R.$$

Something strange happens here.

We shall notice that $a = da_0$, $b = db_0$. So for any nonzero and co-prime elements b, d there exist p, q such that

$$aq + bp = (a, b) = d$$

and $(p, d) = 1$.

As an obvious corollary from this proposition we obtain the following result.

Theorem 6.11. *Let R be such commutative Bezout domain for any nonzero element $a \in R$* $\mathrm{st.r.}(R_a) = 1$. *Then R is elementary divisors ring.*

Proof. It is sufficient to show that for any $a, b, c \in R$ such that $(a, b, c) = 1$ the matrix

$$A = \begin{pmatrix} c & a \\ 0 & b \end{pmatrix}$$

admits canonical diagonal reduction by Theorem **??**.

Let's consider possible cases.

Case 1. Let $c = 0$, so the matrix A is in fact

$$A = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}.$$

Since the Bezout domain R is an Hermite ring then there is $Q \in GL_2(R)$ such that

$$QA \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence matrix A admits canonical diagonal reduction.

Case 2. Suppose that $c \in U(R)$. Then

$$\begin{pmatrix} c & a \\ 0 & b \end{pmatrix} \begin{pmatrix} c^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$$

and matrix A again admits canonical diagonal reduction.

Case 3. Let $c \neq 0$, $c \in U(R)$. Since R satisfies the conditions of Theorem 6.14 then there are $p, q \in R$ such that

$$aR + bR = (ap + bq)R, \ \ pR + (ap + bq)R = R.$$

Therefore

$$1 = (a, b, c) = ((a, b), c) = (c, ap + bq), \ (p, ap + bq) = 1$$

implies that $(cp, ap + bq) = 1$ and by Theorem 3.6 matrix A admits canonical diagonal reduction. □

Proposition 6.15. *Let R a commutative Bezout domain and $a \in R \setminus \{0\}$. Then $J(R_a) \neq 0$.*

Proof. Let M be a maximal ideal of R_a such that a doesn't belong to M. Then $M + aR_a = R_a$, hence there are the elements $m \in M$ and $\frac{r}{s} \in R_a$ such that

$$m + a\frac{r}{s} = 1.$$

Simplifying the latter expression we obtain that $ms + ar = s$.

Let's consider $(m, a) = n$. If n doesn't belong to $U(R)$, then

$$n(m_0 s + a_0 r) = s, \ m = nm_0, \ a = na_0.$$

As n is a divisor of $s \in S_a$ and by Proposition 6.12 S_a is a multiplicatively closed set then $n \in S_a$, i.e. $(n, a) = 1$. Hence $mR + aR = nR + aR = R$ and $m \in U(R_a)$. The latter makes impossible to choose $M \in \mathrm{mspec}(R_A)$ such that $M + aR_a = R_a$. So, a belongs to all maximal ideals of R_a. □

Proposition 6.16. *Let R a commutative Bezout domain and $a \in R \setminus \{0\}$. Then a is a coadequate element of R_a.*

Proof. As in R_a only units are coprime to a then any non invertible element b factors as $b = 1 \cdot b$, where $1R_a + aR_a = R_a$, and for any element $b' \notin U(R_a)$ such that $b' \mid b$ one can check that $b'R_a + aR_a \neq R_a$. □

Theorem 6.12. *Let R be a commutative Bezout domain such that $J(R_a) = aR_a$. Then* st.r.$(R_a) = 1$ *and R is an elementary divisors ring.*

Proof. Consider the quotient ring R_a/aR_a. Since $J(R_a) = aR_a$ then Jacobson radical of this quotient ring is zero and by [????????] any element of R_a/aR_a either is zero divisor or is invertible. Since Jacobson radical of R_a/aR_a is zero then it has no nilpotent elements, i.e. is reduced. Therefore it is possible only if R_a/aR_a is a zero-dimensional ring. So st.r.$(R_a/aR_a) = 1$ and hence R_a is a ring of the almost stable range 1 which has nonzero Jacobson radical.

By Theorem 6.1 st.r.$(R_a) = 1$ and by Theorem 6.11 R is an elementary divisor ring.

Theorem 6.13. *Let R be a commutative Bezout domain such that for any nonzero and non invertible element $a \in R$ the localization R_a is an adequate ring. Then R is an elementary divisor ring.*

Proof. By Proposition 6.15 R_a is an adequate domain whose Jacobson radical is nonzero. By Theorem 4.5 st.r.$(R_a) = 1$. Applying Theorem 6.11 we obtain the desired result. □

Proposition 6.17. *Let R be a commutative Bezout domain and $a \in R \setminus \{0\}$. Then*

1. if $\overline{b}$ isn't a zero divisor in $R_a/\mathrm{rad}(aR_a)$ then b divides a;

2. if a is an adequate element of R_a then st.r.$(R_a) = 1$;

3. if a is not adequate in R_a then $\mathrm{rad}(R_a/aR_a) \neq 0$.

Proof. For the first part suppose that $\overline{b}$ is not a zero divisor in $R_a/\mathrm{rad}(aR_a)$. Since R is a Bezout domain then there are $d, a_0, b_0 \in R$ such that

$$aR + bR = dR,\ a = a_0 d,\ b = b_0 d.$$

Therefore $\overline{ba_0} = \overline{b_0 a} = \overline{0}$ and as $\overline{b}$ is not a zero divisor then $\overline{a_0} = \overline{0}$. Hence there is a natural number n such that $a_0^n = at$. As R is a domain then $a_0^{n-1} = dt$. Since

$(a_0, b_0) = 1$ then $(a_0^{n-1}, b_0) = 1$. As we have shown $d \mid a_0^{n-1}$ so $(d, b_0) = 1$. Therefore $(a_0, b_0) = 1$ and $(d, b_0) = 1$ imply that $(a, b_0) = 1$. Thus b_0 is invertible in R and b is a divisor of a.

For the second part notice that element $a \in J(R_a)$ is adequate and by Corollary 4.1 st.r.$(R_a) = 1$.

Finally, suppose that a is not an adequate element of R_a. If $a = xy$ always provides that $xR_a + yR_a = R_a$ then for any element $b \neq 0$ in R_a the expression $a = a_0(a, b)$ implies that $R_a = a_0R_a + aR_a + bR_a = a_0R_a + bR_a$. So the decomposition $a = a_0(a, b)$ yields ${}_a\mathrm{A}_b$ that contradicts the assumption.

Therefore there are $b, c \in R$ such that $a = bc$ and $(b, c) = d \notin U(R_a)$. Let $b = b_0d, c = c_0d$ for some $b_0, c_0 \in R_a$. Then

$$(bc_0)^2 = bc_0db_0c_0 = bcb_0c_0 = ab_0c_0 \in aR_a.$$

The latter means that $\overline{bc_0} \in \mathrm{rad}(R_a/aR_a)$. If $\overline{bc_0} = \overline{0}$ then there is $t \in R_a$ such that $bc_0 = at$. So $bc_0(1 - dt) = 0$ and $dt = 1$ that contradicts the fact that $d \notin U(R_a)$. Thus $\mathrm{rad}(R_a/aR_a)$ is nonzero. □

6.3 Full matrices over elementary divisor rings

In this section we are going to prove that the class of 2 full matrices over arbitrary elementary divisor ring is of stable range one.

One of the most important problems in determining the connection between the stable range of a given ring and stable range of matrix ring over it was proved in [68]. It states that the stable range of matrix ring of order n over the ring of stable range r equals $1 + [\frac{r-1}{n}]$, where $[m]$ is the least integer greater than or equal m.

It is worth to note that the stable range of a ring R equals 1 if and only if the stable range of any matrix ring over R equals 1. At the same time in [56, 92] there are described some Bezout domains such that class of full matrices over them is of the stable range 1. We are going to establish an analogous result for the elementary divisor rings. As a consequence we also obtain a right (left) divisible chain for two full matrices of order 2 over elementary divisor rings.

Recall that by R_n we denote the matrix ring of order n over a ring R.

Definition 6.6. We say that a matrix $A \in R_n$ is *full* if $R_nAR_n = R_n$. We denote by $F(R_n)$ the class of all full matrices in R_n.

We establish a criterion of left coprimeness of lower triangular 2 matrices.

Theorem 6.14. *Let R be a commutative Bezout ring of the stable range 2. Then matrices*

$$A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} x & 0 \\ y & z \end{pmatrix}$$

are left coprime if and only if $aR + xR = R$ and $cR + zR + (ay - bx)R = R$.

Proof. Suppose that there are matrices $U, V \in R_2$ such that $AU + BV = E$, where E is the identity matrix. Suppose

$$U = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}, \quad V = \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}.$$

Then the equality

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} + \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

implies that $au_1 + xv_1 = 1$, so $aR + xR = R$.

Since the determinant of $AU + BV = E$ equals 1 we obtain that $cR + zR + (ay - bx)R = R$ and the necessity is proved.

Conversely suppose that $aR + xR = R, \quad cR + zR + (ay - bx)R = R$ holds for A and B. Let's consider a matrix

$$C = \begin{pmatrix} a & 0 & x & 0 \\ b & c & y & z \end{pmatrix}.$$

Since $aR + xR = R$ there exist elements $u, v \in R$, such that $au + xv = 1$. Then

$$\begin{pmatrix} a & 0 & x & 0 \\ b & c & y & z \end{pmatrix} \begin{pmatrix} u & 0 & -x & 0 \\ 0 & 1 & 0 & 0 \\ v & 0 & a & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ bu + yv & c & ay - bx & z \end{pmatrix}$$

and using elementary row transformations with the right hand matrix we obtain

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & ay - bx & z \end{pmatrix} = B.$$

Since $cR + zR + (ay - bx)R = R$ and R is a Bezout ring of stable range 2 then according to [81] there exists an invertible 3 matrix P over R such that

$$(c, ay - bx, z)P = (1, 0, 0).$$

Hence

$$B\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & & & \\ 0 & & P & \\ 0 & & & \end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}.$$

In other words we have proved that for the matrix C there exists an invertible 4 matrix Q such that

$$CQ = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}.$$

According to [39] the latter means that for any given A, B, there are some matrices $U, V \in R_2$ such that $AU + BV = E$, that is A, B are left coprime. Theorem is proved. □

We say that a matrix $A \in R_2$ admits a diagonal reduction if there exist invertible matrices $P, Q \in R_2$ such that

$$PAQ = \begin{pmatrix}\varepsilon_1 & 0\\ 0 & \varepsilon_2\end{pmatrix},$$

where $\varepsilon_2 R \subset \varepsilon_1 R$ for any elements $\varepsilon_1, \varepsilon_2 \in R$. If over a ring R any matrix admits a diagonal reduction then the ring is called an elementary divisor ring. We note that if R is a commutative Bezout ring of stable range 2 and $A \in R_2$ is a full matrix which admits a canonical diagonal reduction over R, then $PAQ = \begin{pmatrix}1 & 0\\ 0 & \varepsilon\end{pmatrix}$ for some invertible matrices $P, Q \in R_2$ and element $\varepsilon \in R$.

Theorem 6.15. *Let R be a commutative Bezout ring of stable range 2. Let $A, B \in F(R_2)$ be such that $AR_2 + BR_2 = R_2$. Suppose that B admits canonical diagonal reduction. Then there exists a full matrix $T \in F(R_2)$ such that $A + BT$ is an invertible matrix.*

Proof. Since B admits canonical diagonal reduction then due to the restriction on the ring R we obtain the existence of invertible matrices $P, U, Q \in R_2$ such that

$$PAU = \begin{pmatrix}a & 0\\ b & c\end{pmatrix}, \quad PBQ = \begin{pmatrix}1 & 0\\ 0 & y\end{pmatrix}.$$

Since $AR_2 + BR_2 = R_2$, there exist $C, D \in R_2$ such that $AC + BD = E$, and hence $PAU(U^{-1}C) + PBQ(Q^{-1}D) = P$, that is the matrices $\begin{pmatrix}a & 0\\ b & c\end{pmatrix}$ and $\begin{pmatrix}1 & 0\\ 0 & y\end{pmatrix}$ are left coprime.

By Theorem 6.14, we conclude that $bR+cR+yR=R$. Since R is the ring of stable range 2 then, there exist elements $\alpha,\beta\in R_2$ such that $(b+y\alpha)R+(c+y\beta)R=R$. Hence there are $n,m\in R$ such that $(b+y\alpha)n+(c+y\beta)m=1$. Moreover, by [78] the elements α,β can be chosen such that $\alpha R+\beta R=R$. Then

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}+\begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix}\begin{pmatrix} m-a & -n \\ \alpha & \beta \end{pmatrix}=\begin{pmatrix} m & -n \\ b+y\alpha & c+y\beta \end{pmatrix}$$

is an invertible matrix.

Since $\alpha R+\beta R=R$ then $\begin{pmatrix} m-a & -n \\ \alpha & \beta \end{pmatrix}$ is a full matrix. □

As an obvious consequence of Theorem 6.15, in the case of elementary divisor rings we obtain the following result.

Theorem 6.16. *Let R be a commutative elementary divisor ring. If $A,B\in F(R_2)$ and $AR_2+BR_2=R_2$, then there exists full matrix $T\in F(R_2)$ such that $A+BT$ is an invertible matrix.*

Moreover, we get the following result.

Theorem 6.17. *Let R be a commutative elementary divisor ring. Then for any full matrices $A,B\in F(R_2)$ there exist full matrices $Q_1,Q_2,P\in R_2$ such that $A=BQ_1+P$, $B=PQ_2$.*

Proof. First, note that R_2 is an elementary divisor ring [39], and hence R_2 is a Bezout ring of stable range 2. Let $A,B\in F(R_2)$ and $AR_2+BR_2=DR_2$ for some matrix $D\in R_2$. Hence $A=DA_0$, $B=DB_0$ and $AU+BV=E$ for some matrices $A_0,B_0,U,V\in R_2$.

Then $D(E-A_0U-B_0V)=0$ and $A_0U+B_0V+C=0$ for any matrix $C\in R_2$ such that $DC=0$.

Since R_2 is a ring of stable range 2 then there exist matrices $X,Y\in R_2$ such that

$$(A_0+CX)R_2+(B_0+CY)R_2=R_2.$$

Denote $A_0+CX=A_1$, $B_0+CY=B_1$. As $DC=0$, then $DA_1=A$, $DB_1=B$. Since $A\in F(R_2)$ and $B\in F(R_2)$ and $DA_1=A$, $DB_1=B$, then is clear that $A_1\in F(R_2)$, $B_1\in F(R_2)$. Since $A_1R_2+B_1R_2=R_2$, then by Theorem 6.16 we obtain that $A_1+B_1T=S$ is an invertible matrix for some full matrix T. Hence

$$A_1=B_1(-T)+S,\quad B_1=S(S^{-1}B_1).$$

Then

$$A = B(-T) + DS, \quad B = DS(S^{-1}B_1).$$

Taking $Q_1 = -T$, $Q_2 = S^{-1}B_1$, $P = DS$ and noticing that $Q_1 \in F(R_2)$, $Q_2 \in F(R_2)$, $P \in F(R_2)$ (as appropriate divisors of full matrices) we finished the proof. □

6.4 Sharp Bezout domains

Definition 6.7. Let R be an integral domain and K is its field of fractions. By *an overring* of R we mean any domain between R and K. By the *ring of fractions* of R we mean an overring of R of the form R_S for some nonempty multiplicative set S contained in $R \setminus \{0\}$. We say that R_S is *prime quotient ring* of R if $S = R \setminus P$ for some proper prime ideal P of R and following the notation in **[ ]** we write $R_P = R_S$ in this case.

We denote by $\mathrm{mspec}(R)$ the set of maximal ideals of integral domain R. Gilmer **[ ]** introduced the notion of a sharp domain via "property (#)".

Definition 6.8. We say that a domain R has a *property (#)* if for any two distinct subsets M and N of $\mathrm{mspec}(R)$ we have

$$\bigcap_{P \in M} R_P \neq \bigcap_{P \in N} R_P.$$

Domain R is called a *sharp domain* if each overring of R has property (#). It is said that R has a *QR-property* if each overring of R is a ring of fractions of R **[ ]**.

A characterization of Bezout domains satisfying property (#) is the following result.

Theorem 6.18. **[ ]** *For a Bezout domain R the following conditions are equivalent:*

1. *R satisfies property (#);*

2. *for any* $M \in \mathrm{mspec}(R)$ *there exists some principal ideal aR such that M is a unique maximal ideal of R containing aR.*

Obviously **[ ]**, every Bezout domain with finitely many maximal ideals has property (#). We are going to study the question: when each overring of R has property (#). The following theorem characterizes sharp Bezout domains.

Theorem 6.19. **[ ]** *The following conditions for a domain R are equivalent:*

1. R is a sharp Bezout domain;

2. for each prime ideal P of R there exists a principal ideal $aR \subseteq P$ such that each maximal ideal of R containing aR also contains P.

Let R be a Bezout domain. We denote by $S = S(R)$ the set of all adequate elements of R. Since $1 \in R$ the set S in nonempty. Since S is a saturated multiplicatively closed set, we can consider the localization of R with denominators from S, i.e. the ring of fractions R_S.

Theorem 6.20. *A Bezout domain is an elementary divisor ring if and only if R_S is an elementary divisor ring.*

Proof. By Theorem 3.4 it is sufficient to prove the statement for matrices

$$A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix},$$

where $aR + bR + cR = R$. Suppose that R_S is an elementary divisor ring. Then for elements $a, b, c \in R_S \cap R$ one can find elements $ps^{-1}, qs^{-1} \in R_S$, where $s \in S$, such that

$$(aps^{-1} + bqs^{-1})R_S + cqs^{-1}R_S = R_S.$$

Then there are elements $r, t, k, l \in R$, $s \in S$ such that

$$(ak + bl)s + clt = s \in S.$$

Thus, there exists an equivalent to A matrix B of form

$$B = \begin{pmatrix} z & 0 \\ x & y \end{pmatrix},$$

where z divides s (so $z \in S$), as well as $xR + yR + zR = R$. Since z is an adequate element, it is easy to see that the matrix A admits canonical diagonal reduction. Indeed, according to the fact that z is adequate there are elements $r, s \in R$ such that $z = rs$, where $rR + yR = R$ and $s'R + yR \neq R$ for arbitrary nonunit divisor s' of the element $s \in R$. We are going to prove that $(y + rx)R + rzR = R$. Otherwise, if $(y + rx)R + rzR = hR \neq R$ then $rzR \subset hR$. If $hR + rR = \delta R \neq R$ then $(y + rx)R \subset \delta R$ and therefore $yR \subset \delta R$ which is impossible since $rR \subset \delta R$ and $rR + yR = R$. Therefore $sR \subset hR$. Hence, according to the definition of s, we have $hR + yR = \delta R \neq R$. Then $(z + rx)R \subset \delta R$ and $zR \subset \delta R$. Since $\delta R + rR = rR$ then $xR \subset \delta R$ that is impossible as $xR + yR + zR = R$ and $\delta R \neq R$. Therefore

$$\begin{pmatrix} z & 0 \\ x & y \end{pmatrix} \begin{pmatrix} r & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} zr & z \\ xr+y & x \end{pmatrix} = C$$

Since $rzR + (xr + y)R = R$ and R is a Bezout domain then the matrix C and, moreover, the matrix B admits canonical diagonal reduction. Therefore, R is an elementary divisor ring.

Conversely, suppose that R is an elementary divisor ring. We need to show that R_S is also an elementary divisor ring. Let $as^{-1}, bs^{-1}, cs^{-1}$ be any elements form R_S such that

$$as^{-1}R_S + bs^{-1}R_S + cs^{-1}R_S = R_S.$$

Then $aR + bR + cR = dR$ for some element $d \in S$. Let $a = a_1 d$, $b = b_1 d$, $c = c_1 d$, for some elements $a_1, b_1, c_1 \in R$ such that $a_1 R + b_1 R + c_1 R = R$. Since R is an elementary divisor ring by Theorem 3.6 there are elements $u, v, p, q \in R$ such that

$$(a_1 p + b_1 q)u + c_1 qv = 1.$$

Then

$$(aps^{-1} + bqs^{-1})R_S + cqs^{-1}R_S = R_S.$$

By the Theorem 3.6, R_S is an elementary divisor ring. Theorem is proved. □

Let R be a commutative Bezout domain, and $S = S(R)$ be a set of all adequate elements of R. Since $S = S(R)$ is a saturated multiplicatively closed set, we can construct a D-chain.

By Theorem 6.20 and using D-chain of Bezout domain we can conclude that the problem of being commutative Bezout domain an elementary divisor ring is reduced to the case of domain where only adequate elements are the units.

Theorem 6.21. *Every sharp Bezout domain R is an elementary divisor ring.*

Proof. Let R be a sharp Bezout domain and $M \in \mathrm{mspec}(R)$. By Theorem 6.18 there exists a principal ideal aR such that M is the unique maximal ideal of R containing an ideal aR. Take any $b \in R$. If $b \notin M$ then $aR + bR = R$. If $b \in M$ then $a = 1\dot{a}$ and for each nonunit divisor s' of a we have $s'R + bR \neq R$. Then a is an adequate element of R. Since in a sharp Bezout domain D-chain exists and is nontrivial, and since Bezout domain has QR-property **[ ]** we obtain that R is an elementary divisor domain. Theorem is proved. □

Theorem 6.22. *A sharp Bezout domain R is an adequate domain if and only if each nonzero prime ideal of R is contained in a unique maximal ideal of R.*

Proof. By **[ ]** a sharp Bezout domain R in which every nonzero prime ideal of R contained in a unique maximal ideal is a semilocal Bezout domain and by **[ ]** R is an adequate domain. Since every nonzero prime ideal of an adequate ring is contained in a unique maximal ideal, the theorem is proved. □

We are going to finish with an example related to the previously established results .

Example 6.2. Let R be a ring defined in Example 4.3, i.e.

$$R = \{z_0 + a_1x + a_2x^2 + \dots | z_0 \in \mathbb{Z}, a_i \in \mathbb{Q}\}.$$

Then R is a two-dimensional sharp Bezout domain with a nontrivial D-chain.

Remark 6.1. In **[ ]** it is presented an example of a Bezout domain D which has only finitely many minimal prime ideals over each principal ideal of R, but R does not have property (#). This example is an example of a Bezout domain **[ ]**, which is an elementary divisor domain, which does not have property (#).