Chapter 7
Diagonalization over noncommutative rings

7.1 Simple Ore and Bezout rings

It is known [84] that a simple Bezout domain is an elementary divisor domain if and only if it is 2-simple. In this section we are going to prove that over 2-simple Ore domain of stable range 1 every matrix that is not a zero divisor is equivalent to a canonical diagonal matrix.

Definition 7.1. Recall, that a ring R is said to be *simple* provided if I is a two-sided ideal of R then $I = R$ or $I = 0$.

Remark 7.1. Suppose that R is a simple ring and $a \in R \setminus \{0\}$. Hence $RaR = R$ so there are $u_1, u_2, ..., u_n; v_1, v_2, ..., v_n \in R$ such that

$$u_1 a v_1 + u_2 a v_2 + ... + u_n a v_n = 1.$$

Therefore for each nonzero element of simple ring one can find such n as described above. This motivates us to consider a case when such number is common for all elements $a \in R$.

Definition 7.2. [52] If for any nonzero element $a \in R$ of a simple ring R there is some $n \in \mathbb{N}$ such that

$$u_1 a v_1 + ... + u_n a v_n = 1$$

and n is the least possible then R is called $n-$*simple ring*.

In the following we will mainly consider 2-simple domains R, i.e. a domains R such that for any nonzero element $a \in R$ there are $u_1, u_2; v_1, v_2 \in R$ such that

$$u_1 a v_1 + u_2 a v_2 = 1.$$

One of the important cases of 2-simple rings are known as Kazimirsky rings in the name of P. S. Kazimirsky who was the first who considered such class of rings [40].

Definition 7.3. A ring R is called *a Kazimirsky ring* if for any pair of nonzero elements $a,b \in R$ there are $x,y \in R$ and $u \in U(R)$ such that $ax+uby=1$.

Example 7.1. [52, 40] A ring of $n-$derivations is a Kazimirsky domain.

Lemma 7.1. *Let $a_1,\dots,a_n \in R$ be nonzero elements of n-simple domain. Then there are $u_1,\dots,u_n; v_1,\dots,v_n \in R$ such that*

$$u_1 a v_1 + \ldots + u_n a_n v_n = 1.$$

Proof. Since R is a domain and elements $a_1,\dots,a_n \in R$ are nonzero then so is their product $a=a_1a_2\dots a_n$ and therefore $RaR=R$. Using the fact that R is n-simple one can find some $x_1,\dots,x_n; y_1,\dots,y_n \in R$ such that

$$x_1 a y_1 + \ldots + x_n a y_n = 1.$$

If we substitute $a=a_1a_2\dots a_n$ then

$$x_1 a_1 (a_2 \dots a_n y_1) + \ldots + (x_n a_1 a_2 \dots a_{n-1}) a_n y_n = 1$$

that proves the statement. □

As a simple corollary in case of 2-simple stable range 1 ring we obtain

Proposition 7.1. *Every* 2*-simple domain of stable range 1 is a Kazimirsky ring.*

Proof. Let $a,b \in R$ be any two nonzero elements. Since $ab \neq 0$ then by Lemma 7.1 there are $x_1,x_2,y_1,y_2 \in R$ such that $x_1ay_1+x_2by_2=1$. Now, as st.r.$(R)=1$ then $x_1aR+(x_2a)bR=R$ implies $x_1a+(x_2a)bz \in U(R)$ for some $z \in R$. Hence $Ra+Rbz=R$ and again there is $t \in R$ such that $a+tbz \in U(R)$ by the stable range condition.

As $aR+tR=R$ then $as+t=w$ for some $s \in R,\ w \in U(R)$. Substituting $t=w-as$ in $a+tbz \in U(R)$ we obtain

$$a+tbz = a+(w-as)bz = a(1-sbz)+wbz \in U(R)$$

as was desired. □

Definition 7.4. [10] A domain R is called a *right (left) Ore domain* provided for any nonzero $a,b \in R$

$$aR \cap bR \neq 0,\ (Ra \cap Rb \neq 0).$$

We say that R is an *Ore domain* if it is both right and left Ore domain.

A domain in Example 7.1 is an Ore domain.

Proposition 7.2. *[63] Every right Bezout or right noetherian domain is a right Ore domain.*

Proof. Suppose R is a right Bezout domain such that there are nonzero $a,b \in R$ and $aR \cap bR = 0$. Let $aR + bR = dR$ and $b = db_0$.

Since R is a domain then $aR \cong R/r(a) = R$ and

$$R \cong aR = aR/(aR \cap bR) \cong (aR + bR)/bR = dR/db_0R \cong R/b_0R.$$

Hence there is a split exact sequence:

$$0 \longrightarrow b_0R \longrightarrow R \longrightarrow R \longrightarrow 0,$$

so $b_0R \oplus R \cong R$. Hence b_0R is a direct summand of R and this is possible if and only if $b_0R = eR$ for some idempotent $e \in R$. Since R is a domain then either $b_0R = 0$ or $b_0R = R$. In the latter case $bR = dR \supset aR$ that contradicts the fact that $a \neq 0$ and $aR \cap bR = 0$. Thus $b_0 = 0$ and then $d = 0$. So $a = b = 0$ that is again a contradiction.

If R is a right noetherian domain then for any nonzero elements $a,b \in R$ we consider right ideals

$$I_n = bR + abR + \ldots + a^n bR.$$

By the right noetherian condition there is a smallest n such that $I_n = I_{n+1}$. Thus

$$a^{n+1}b = bc_0 + abc_1 + \ldots + a^n bc_n \Rightarrow bc_0 = a(a^n b - bc_1 - \ldots - a^{n-1}bc_n) \neq 0,$$

where the minimality of n implies that the bracket is nonzero. Thus $aR \cap bR \neq 0$. □

Also there is a useful application of Ore domain condition for 2×2 matrices.

Lemma 7.2. *Suppose $A \in R_2$ is not a zero divisor and R is a right Ore domain. Then there is a matrix $T \in R_2$ such that AT is a diagonal matrix.*

Proof. Let A be not a zero divisor. Then any of its columns or rows is nonzero. Without loss of generality suppose that $a_{11} \neq 0$. If $a_{12} = a_{21} = 0$ then $a_{22} \neq 0$ and A is already diagonal. So, let $a_{21} = 0$ and a_{12} and a_{22} are both nonzero. Therefore $a_{11}R \cap a_{12}R \neq 0$ and there are $x,y \in R$ such that $a_{11}x = -a_{12}y$. Hence

$$\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix} = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22}y \end{pmatrix}$$

is a diagonal matrix. Similarly we deal with the lower triangular matrix case.

If all entries of A are nonzero then there are $x, y, u, v \in R$ such that $a_{11}x = -a_{12}y$ and $a_{21}u = -a_{22}v$. Therefore

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} u & x \\ v & y \end{pmatrix} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

is a diagonal matrix.

Theorem 7.1. *Simple Bezout domain is an elementary divisor ring if and only if R is 2-simple.*

Proof. Suppose that R is a simple elementary divisor domain and a be arbitrary nonzero element of R. Then the matrix $A = \mathrm{diag} a, a$ admits canonical diagonal reduction, i.e. there are $P, Q \in GL_2(R)$ and $z, b \in R$ such that

$$AP = Q\mathrm{diag} z, b,$$

where $RbR \subseteq zR \cap Rz$. Since R is a simple then either $b = 0$ or $z \in U(R)$. In case when $b = 0$ we obtain that $ap_{12} = ap_{22} = 0$. But R is a domain and $a \neq 0$, so $p_{12} = p_{22} = 0$. This contradicts the fact that $P \in GL_2(R)$.

Therefore $z \in U(R)$ and without loss of generality ew can assume that $z = 1$.

As Q is invertible then $Rq_{11} + Rq_{21} = R$ implies that $uq_{11} + vq_{21} = 1$ for some elements $u, v \in R$. Since $AP = Q\mathrm{diag} 1, b$ then $ap_{11} = q_{11},\ ap_{21} = q_{21}$ so

$$uap_{11} + vap_{21} = 1,$$

i.e. R is 2-simple.

Conversely, suppose that R is a 2-simple Bezout domain and $A \in M_2(R)$. Since Bezout domain is an Hermite ring then by Theorem [???????] we can assume that A is a triangular matrix.

If matrix A is of type

$$\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

for some $a, b \in R$ then it is equivalent to $\mathrm{diag} d, 0$ for some element d in R since R is an Hermite ring.

Now suppose that there are $a, b, c \in R \setminus \{0\}$ such that

$$A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}.$$

Since any Bezout domain is an Ore domain by Proposition 7.2 then by Lemma 7.2 there is $T \in R_2$ such that

$$AT = \begin{pmatrix} g & 0 \\ 0 & f \end{pmatrix}$$

for some $g, f \in R \setminus \{0\}$. By Lemma 7.1 there are some elements $u_1, u_2, v_1, v_2 \in R$ such that

$$\begin{pmatrix} u_1 & u_2 \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & f \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 1.$$

Since $u_1R + u_2R = R$ and R is an Hermite ring then there is an invertible matrix S such that

$$\begin{pmatrix} u_1 & u_2 \end{pmatrix} S = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

Therefore $\begin{pmatrix} u_1 & u_2 \end{pmatrix}$ is a first row of invertible matrix $P = S^{-1}$. If we consider a column

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = T \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

the equality

$$\begin{pmatrix} u_1 & u_2 \end{pmatrix} AT \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \end{pmatrix} A \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 1.$$

implies that $Rw_1 + Rw_2 = R$ and similarly a column $\begin{pmatrix} w_1 & w_2 \end{pmatrix}^T$ is completable to invertible matrix Q. As a result we have that

$$PAQ = \begin{pmatrix} 1 & * \\ * & * \end{pmatrix}$$

and the latter matrix admits canonical diagonal reduction. Theorem is proved. □

Corollary 7.1. *Simple Bezout domain of stable range 1 is an elementary divisor ring if and only if R is a Kazimirsky ring.*

Proof. The necessity follows from Theorem 7.1 and Proposition 7.1. Since every Kazimirsky ring is a 2-simple ring by Theorem 7.1 we obtain the sufficiency condition.

If we take only the Ore domain condition instead of Bezout one we can obtain the following result using the Proposition 7.1.

Proposition 7.3. *Let R be a Ore and Kazimirsky domain. Then for any 2×2 matrix A that is not a zero divisor there are $e, f \in R$, $u \in U(R)$ such that $Re + Rf = R$ and*

$$(1, u)A \begin{pmatrix} e \\ f \end{pmatrix} = 1.$$

Proof. Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ Since R is an Ore domain and A is not a zero divisor then by Lemma 7.2 there is a matrix $D \in R_2$ (also not zero divisor) such that

$$AD = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix},$$

where $d_1 \neq 0, d_2 \neq 0$.

Considering the pair of elements d_1, d_2 and using the definition of Kazimirsky domain there are $c, d \in R$ and $u \in U(R)$ such that $d_1 c + u d_2 d = 1$. Hence

$$(1\ u) AD \begin{pmatrix} c \\ d \end{pmatrix} = (1\ u) \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = 1.$$

If we take

$$D \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix},$$

then we obtain the desired statement. □

For a technical purposes in the proofs below we need a few lemmas.

Lemma 7.3. *[29] For any ring R a matrix $A = \begin{pmatrix} a & -t \\ b & 1 \end{pmatrix} \in GL_2(R)$ if and only if $a + tb \in R$.*

Proof. When $b = 0$ the statement is trivially verified. If $b \neq 0$ then

$$\begin{pmatrix} a & -t \\ b & 1 \end{pmatrix} \in GL_2(R) \text{if and only if} \begin{pmatrix} a & -t \\ b & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} = \begin{pmatrix} a + tb & -t \\ 0 & 1 \end{pmatrix} \in GL_2(R)$$

and the statement follows from the above case. □

Using Lemma 7.3 we obtain the following result:

Lemma 7.4. *Any left (right) unimodular row (column) $Ra + Rb = R$ ($aR + bR = R$) over stable range 1 ring R is completable to some invertible matrix*

$$\begin{pmatrix} a & * \\ b & * \end{pmatrix} \quad \left(\begin{pmatrix} a & b \\ * & * \end{pmatrix} \right).$$

Proof. If $Ra + Rb = R$ and st.r.$(R) = 1$ then there is $t \in R$ such that $a + tb \in U(R)$. By Lemma 7.3 given row is completable. □

Lemma 7.5. *For any domain R a column $\begin{pmatrix} a\, b\, 1 \end{pmatrix}^T$ is completable to invertible matrix.*

Proof. The statement follows from the following equalities

$$\begin{pmatrix} a & 1 & 0 \\ b & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -a \\ 0 & 1 & -b \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -a \\ 0 & 1 & -b \end{pmatrix} \begin{pmatrix} a & 1 & 0 \\ b & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Lemma is proved. □

Theorem 7.2. *Let R be an Ore and Kazimirsky domain. Then for any matrix $A \in R_n$, $(n \geq 2)$ that is not a zero divisor there are some invertible matrices P, Q such that*

$$PAQ = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \dots & a & b \\ 0 & 0 & \dots & c & d \end{pmatrix}.$$

Proof. Taking into account Proposition 7.3 we are going to use induction on size of matrix A. Since the case $n = 2$ is straightforward we assume that $n = 3$. Let $n = 3$ and $A = (a_{ij})$ is a 3×3 matrix. Without loss of generality we can assume that a submatrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

of matrix A also is not a zero divisor. By Proposition 7.3 we can obtain the equality

$$\begin{pmatrix} 1\, u\, 0 \end{pmatrix})A \begin{pmatrix} e - e(a_{13} + ua_{23}) \\ f - f(a_{13} + ua_{23}) \\ 1 \end{pmatrix} = 1,$$

for some $u, e, f \in R$. Clearly the row $\begin{pmatrix} 1\, u\, 0 \end{pmatrix})$ is completable to invertible matrix P. On the other hand column $\begin{pmatrix} e - e(a_{13} + ua_{23})\ f - f(a_{13} + ua_{23})\ 1 \end{pmatrix}^T$ is completable too to some invertible matrix Q by Lemma 7.5. Hence matrix

$$PAQ = \begin{pmatrix} 1 & a'_{12} & a'_{13} \\ a'_{21} & a'_{22} & a'_{23} \\ a'_{31} & a'_{32} & a'_{33} \end{pmatrix}.$$

using elementary transformations of rows and columns can be reduced to form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix},$$

hence the statement is proved for $n = 3$.

Consider the case $n > 3$. Suppose that the statement is proved for $n-1 \geq 3$. Without loss of generality [10] we can assume that 3×3 submatrix in the upper left corner of matrix A is not a zero divisor. Due to the proved above statement for $n = 3$ we can find invertible matrices $P_1, Q_1 \in GL_3(R)$ such that

$$P_1 \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} Q_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a'_{22} & a'_{23} \\ 0 & a'_{32} & a'_{33} \end{pmatrix}.$$

If we denote by I_k the $k \times k$ identity matrix then

$$\begin{pmatrix} P_1 & 0 \\ 0 & I_{n-3} \end{pmatrix} A \begin{pmatrix} Q_1 & 0 \\ 0 & I_{n-3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & a'_{14} & \dots & a'_{1n} \\ 0 & a'_{22} & a'_{23} & a'_{24} & \dots & a'_{2n} \\ 0 & a'_{32} & a'_{33} & a'_{34} & \dots & a'_{3n} \\ a'_{41} & a'_{42} & a'_{43} & a'_{44} & \dots & a'_{4n} \\ \dots & \dots & \dots & \dots & \ddots & \dots \\ a'_{n1} & a'_{n2} & a'_{n3} & a'_{n4} & \dots & a'_{nn} \end{pmatrix}.$$

Using elementary transformations of rows and columns we obtain that matrix A is equaivalent to matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & A_1 \end{pmatrix}$$

for some matrix $A_1 \in R_{n-1}$ that is not a zero divisor. By the induction hypothesis there exist $P_2, Q_2 \in GL_{n-1}(R)$ such that

$$\begin{pmatrix} 1 & 0 \\ 0 & P_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & A_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & Q_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & I_{n-3} & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}.$$

Applying the induction we finish the proof. □

Theorem 7.3. *Let R be a 2-simple Ore domain of stable range 1. Then for any matrix $A \in R_n$ that is not a zero divisor there are some invertible matrices $P, Q \in GL_n(R)$ such that*

$$PAQ = \begin{pmatrix} 1\,0 \dots 0\,0 \\ 0\,1 \dots 0\,0 \\ 0\,0 \ddots 0\,0 \\ 0\,0 \dots 1\,0 \\ 0\,0 \dots 0\,\Delta \end{pmatrix}.$$

Proof. Since R is a Kazimirsky domain by Proposition refprop194 then applying Proposition 7.3 to arbitrary 2×2 matrix A that is not a zero divisor we obtain that there are $e, f \in R$ and $u \in U(R)$ such that $Re + Rf = R$ and

$$\begin{pmatrix} 1\; u \end{pmatrix} A \begin{pmatrix} e \\ f \end{pmatrix} = 1.$$

Since R is a stable range 1 ring then by Lemma 7.4 $(1, u)$ and $(e, f)^T$ are completable to invertible matrices $P, Q \in GL_2(R)$ respectively. Hence

$$PAQ = \begin{pmatrix} 1 * \\ * * \end{pmatrix}.$$

Using elementary transformations PAQ can be reduced to

$$\begin{pmatrix} 1\; 0 \\ 0\; \Delta \end{pmatrix}.$$

The latter means that there are invertible matrices $S, T \in GL_2(R)$ such that

$$SAT = \begin{pmatrix} 1\; 0 \\ 0\; \Delta \end{pmatrix}.$$

If we take arbitrary matrix $A \in R_n$, where $n \geq 2$, then by Theorem 7.2 matrix A is equivalent to

$$\begin{pmatrix} I_{n-2} & 0 \\ 0 & A_1 \end{pmatrix}$$

for some $A_1 \in R_2$ that is not a zero divisor. By the proved above $n = 2$ case we obtain that A_1 is equivalent to diag$1, \Delta$ for some Δ. Hence there are invertible matrices $P, Q \in GL_n(R)$ such that

$$PAQ = \begin{pmatrix} I_{n-1} & 0 \\ 0 & \Delta \end{pmatrix}$$

that was desired. □

Now we are going to deal with the general case of $n-$simple Bezout domain, when $n \geq 3$.

Theorem 7.4. *Let R be an $n-$simple Bezout domain, where $n \geq 3$, and A be an arbitrary $m \times m$ matrix, $m \geq n$. Then there are invertible matrices $P, Q \in GL_m(R)$ such that PAQ is a block-diagonal sum of identity matrix I_r and some triangular $n \times n$ matrices $A_1, ...A_k$:*

$$PAQ = \begin{pmatrix} I_r & 0 & \dots & 0 \\ 0 & A_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_k \end{pmatrix},$$

where $m = nk + r$, $k \in \mathbb{N}$, $0 \leq r \leq n-1$.

Proof. We are going to prove the statement by induction on the number m. Let $m = n$. Since any Bezout domain is an Hermite ring we can assume that A has the form:

$$A = \begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}.$$

We consider all possible cases.

Case 1. Let $a_{11} = 0$, i.e. the matrix A has form

$$A = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}.$$

Since R is an Hermite ring then there is an invertible matrix $Q' \in GL_{n-1}(R)$ such that

$$B = \begin{pmatrix} a_{21} & a_{22} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix} Q'$$

is a triangular matrix. Hence the matrix A is equivalent to matrix

$$\begin{pmatrix} B & O \\ 0 & 0 \end{pmatrix},$$

that was desired.

Case 2. Let $a_{ii} = 0$, where $i > 1$, i.e. the matrix A has form

$$A = \begin{pmatrix} a_{11} & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 & 0 & 0 & \dots & 0 \\ a_{i1} & a_{i2} & \dots & a_{ii-1} & 0 & 0 & \dots & 0 \\ a_{i+11} & a_{i+12} & \dots & 0 & 0 & a_{i+1i+1} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots & \\ a_{n1} & a_{n2} & \dots & a_{ni-1} & a_{ni} & a_{ni+1} & \dots & a_{nn} \end{pmatrix}.$$

Then for the row $(a_{i,1}, a_{i,2}, \dots, a_{i,i-1})$ there exists $Q'' \in GL_{i-1}(R)$ such that

$$(a_{i,1}, a_{i,2}, \dots, a_{i,i-1})Q'' = (a'_{i,1}, 0, \dots, 0).$$

Therefore A is equivalent to the matrix

$$A' = \begin{pmatrix} a_{11} & 0 & \dots & 0 & 0 \\ a_{21} & a_{22} & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \ddots & \vdots \\ a'_{i1} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn-1} & a_{nn} \end{pmatrix}.$$

Rearranging the rows we conclude that A' is equivalent to the matrix

$$\begin{pmatrix} a_{11} & 0 & \dots & 0 \\ a'_{i1} & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}.$$

Since R is an Hermite ring there exists an invertible matrix $P \in GL_2(R)$ such that

$$P \begin{pmatrix} a_{11} \\ a'_{i1} \end{pmatrix} = \begin{pmatrix} 0 \\ a''_{i1} \end{pmatrix},$$

for some $a''_{i1} \in R$. Therefore the matrix A', and hence the matrix A are equivalent to the matrix

$$\begin{pmatrix} B & O \\ 0 & 0 \end{pmatrix},$$

where B is a triangular matrix of order $m = n$ that was desired.

Case 3. Assume that A has the following form (up to matrix equivalence)

$$\begin{pmatrix} a_{11} & 0 & \dots & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ii} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{ni} & \dots & a_{nn} \end{pmatrix},$$

where $a_{11} \neq 0, a_{22} \neq 0, \dots a_{nn} \neq 0$. In this case we are going to show that there exists a matrix T or size n such that

$$AT = \begin{pmatrix} \varepsilon_1 & 0 & \dots & 0 \\ 0 & \varepsilon_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \varepsilon_n \end{pmatrix}$$

for some elements $\varepsilon_1, \varepsilon_2, \dots \varepsilon_n \in R$.

Let $a_{21} \neq 0$. Since any Bezout domain in an Ore domain [21] there are elements $x, y \in R$ such that $a_{21}x = -a_{22}y$. Then

$$\begin{pmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x & 0 & 0 & \dots & 0 \\ y & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} = \begin{pmatrix} a_{11}x & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}.$$

It is clear that $a_{11}x \neq 0$. Thus the case $a_{21} \neq 0$ is reduced to the case $a_{21} = 0$. If we continue this process moving down along the matrix then we get a matrix T such that

$$AT = \begin{pmatrix} \varepsilon_1 & 0 & 0 & \dots & 0 \\ 0 & \varepsilon_2 & 0 & \dots & 0 \\ 0 & 0 & \varepsilon_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \varepsilon_n \end{pmatrix},$$

where $\varepsilon_1 \neq 0, \varepsilon_2 \neq 0, \dots \varepsilon_n \neq 0$.

Since R is $n-$simple then by Lemma 7.1 for elements $\varepsilon_1, \varepsilon_2, \dots \varepsilon_n$ one can find $u_1, u_2, \dots u_n, v_1, v_2 \dots v_n \in R$ such that

$$(u_1 \dots u_n) \begin{pmatrix} \varepsilon_1 & 0 & \dots & 0 \\ 0 & \varepsilon_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \varepsilon_n \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = 1.$$

This yeilds

$$(u_1 \dots u_n) AT \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = 1.$$

Then there are elements $w_1, w_2 \dots w_n \in R$ such that

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

and therefore $Rw_1 + \dots + Rw_n = R$.

Since R is an Hermite ring then by Theorem [????????] st.r.$(R) \leq 2$ and by Theorem [????????] any unimodular row or column of length $l \geq 2+1 = 3$ is completable to invertible matrix of appropriate size. So, one can complete the row $u_1, \dots, u_n)$ and column $v_1, \dots, v_n$ to $U, V \in GL_n(R)$ respectively. Hence

$$UAV = \begin{pmatrix} 1 & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$$

and UAV can be reduced using elementary transformations to the form

$$\begin{pmatrix} 1 & 0 \\ 0 & A' \end{pmatrix}.$$

Summarizing Cases 1,2 and 3 we conclude that the base of induction is proved. The induction on the size of the matrix completes the proof of theorem. □

7.2 Idempotent matrix diagonalization

In this section we will show that among all von Neumann regular rings only unit-regular rings are idempotently diagonalizable.

Let R be a associative ring with unit element and $1 \neq 0$.

Definition 7.5. [36] A ring R is called an *idempotently diagonalizable* if for every square matrix A of order n over R there exist invertible matrices $P,Q \in GL_n(R)$ such that

1. $PAQ = D$ is a diagonal matrix;

2. $D^2 = D$.

Example 7.2. A commutative von Neumann regular ring is an obvious example of idempotently diagonalizable ring.

Theorem 7.5. *In a class of von Neumann regular rings only unit-regular rings are idempotently diagonalizable.*

Proof. Let R be an unit-regular ring. By [36] for any square matrix A of order n over R there exist invertible matrices $P,Q \in GL_n(R)$ such that

$$\begin{pmatrix} d_1 & 0 & \dots & 0 & 0 & \dots 0 \\ 0 & d_2 & \dots & 0 & 0 & \dots 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ 0 & 0 & \dots & d_r & 0 & \dots 0 \\ 0 & 0 & \dots & 0 & 0 & \dots 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots 0 \end{pmatrix}$$

Since R is unit-regular then for any element $a \in R$ there exist unit element $u \in U(R)$ and an idempotent $e = e^2 \in R$ such that $au = e$. Let $d_i u_i = e_i$, $i = 1,2,\dots,n$, where $u_i \in U(R)$ and $e_i^2 = e_i$.

Let

$$D = \begin{pmatrix} e_1 & 0 & \dots & 0 & 0 & \dots 0 \\ 0 & e_2 & \dots & 0 & 0 & \dots 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ 0 & 0 & \dots & e_r & 0 & \dots 0 \\ 0 & 0 & \dots & 0 & 0 & \dots 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots 0 \end{pmatrix} U,$$

where

$$U = \begin{pmatrix} u_1 & 0 & \dots & 0 & 0 & \dots 0 \\ 0 & u_2 & \dots & 0 & 0 & \dots 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ 0 & 0 & \dots & u_r & 0 & \dots 0 \\ 0 & 0 & \dots & 0 & 0 & \dots 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots 0 \end{pmatrix} \in GL_n(R).$$

Then

$$PAQU = \begin{pmatrix} e_1 & 0 & \dots & 0 & 0 & \dots 0 \\ 0 & e_2 & \dots & 0 & 0 & \dots 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ 0 & 0 & \dots & e_r & 0 & \dots 0 \\ 0 & 0 & \dots & 0 & 0 & \dots 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots 0 \end{pmatrix} = E.$$

and $E^2 = E$ that R is of idempotently diagonalizable ring.

Let R be an idempotently diagonalizable von Neumann regular ring, i.e. for any square matrix A over R there exist invertible matrices P, Q such that $PAQ = E$ is a diagonal matrix such that $E^2 = E$.

Then $A = P^{-1}EQ^{-1}$ and $AQPA = P^{-1}EQ^{-1}QPP^{-1}EQ^{-1} = P^{-1}E^2Q^{-1} = P^{-1}EQ^{-1} = A$.

Therefore R_r is a unit-regular ring and by **[?]** R is a unit-regular ring. Theorem is proved. □

7.3 Distributive Bezout rings and Dubrovin condition

Lam and Dugas [42] asked a question: is every right quasi-duo ring a left one and vice versa? So the question is can one find a one-sided quasi-duo ring that is not two-sided quasi-duo one. In this section we introduce a notion of weak stable range 1 ring in order to show that every one-sided quasi-duo ring of weak stable range 1 is in fact two-sided one, and Bezout domain of weak stable range 1 is an elementary divisor ring if and only if it is duo-domain.

We start with recalling the necessary definitions as well as basic properties of considered rings.

Definition 7.6. [42] A ring R is called *a right (left) quasi-duo ring* if every right (left) maximal ideal of R is two-sided. A ring that is both right and left quasi-duo ring is called quasi-duo ring.

Theorem 7.6. *For any ring R the following conditions are equivalent:*

- *R is a right (left) quasi-duo ring;*
- *for any elements $a,b \in R$ whenever $Ra+Rb=R$ $(aR+bR=R)$ then $aR+bR=R$ $(Ra+Rb=R)$.*

Definition 7.7. [????????] A ring R is said to be *a weak stable range 1 ring* provided whenever $Ra+Rb=R$ for $a,b \in R$ there is an element $t \in R$ such that $a+tb=1$.

Theorem 7.7. *Let R be a right (left) quasi-duo ring of weak stable range 1. Then R is a quasi-duo ring.*

Proof. Consider arbitrary elements $a,b \in R$ such that $aR+bR=R$. Since any weak stable range 1 ring has stable range 1 then there is an element $t \in R$ such that $a+bt=u$, where $u \in U(R)$. The latter means that $Ra+Rt=R$.

By the definition of weak stable range there is an element $x \in R$ such that $xa+t=1$. Hence $t=1-xa$ implies that $a+b(1-xa)=u$. Therefore

$$a+b-bxa=b+(1-bx)a=u \in U(R)$$

yields $Ra+Rb=R$. By Theorem 7.6 R is a left quasi-duo ring. The opposite case is similar. Theorem is proved. □

Theorem 7.8. *Let R be a Bezout right quasi-duo domain of weak stable range 1. Then R is an elementary divisor ring if and only if R is a duo-ring.*

Proof. For sufficiency suppose that R is a Bezout duo-domain of weak stable range 1. By [????????????] it is sufficient to show that for any elements $a,b,c \in R$ such that $aR+bR+cR=R$ the matrix

$$A=\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$$

admits canonical diagonal reduction.

Suppose that $Ra+Rb=Rd$. Since R is a Bezout domain of weak stable range 1 there is an element $t \in R$ such that $ta+b=d$. Since R is a duo-ring then $dR=Rd$ and $aR+bR=dR$. Hence

$$dR + cR = aR + bR + cR = R$$

and using the weak stable range condition there is an element $s \in R$ such that $d + cs = 1$. Therefore

$$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ ta+b & c \end{pmatrix}$$

$$\begin{pmatrix} a & 0 \\ ta+b & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ ta+b+cs & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 1 & c \end{pmatrix} = B.$$

We conclude that both B and A can be reduced to canonical diagonal form using elementary transformations of rows and columns. So, R is an elementary divisor ring.

To prove the necessity suppose that R is an elementary divisor ring of weak stable range 1. Then for any $a \in R$ there exist $P, Q \in GL_2(R)$ such that

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} P = Q \begin{pmatrix} z & 0 \\ 0 & b \end{pmatrix}$$

where $RbR \subseteq zR \cap Rz$.

From the above matrix equality we obtain that

$$ap_{11} = q_{11}z, ap_{21} = q_{21}z.$$

Since P, Q are invertible and $RbR \subseteq zR \cap Rz$ we get $RaR = RbR$. The invertibility of P implies that $Rp_{11} + Rp_{21} = R$ and by Theorem 7.6 $p_{11}R + p_{21}R = R$, i.e. there are $u, v \in R$ such that $p_{11}u + p_{21}v = 1$. Then $a = a(p_{11}u + p_{21}v) = q_{11}zu + q_{21}zv$ and $a \in RzR$.

Reminding that $RaR = RbR$ we obtain $RaR = zR = Rz$. Since R is a domain then $a = za_0$, $a = a_1z$ for some elements $a_0, a_1 \in R$, $Ra_0R = Ra_1R = R$. Due to Theorem 7.7 and [6] elements a_0, a_1 are invertible, so a is a duo-element and R is a duo-ring.

7.4 When $GL_n(R)$ is closed under transposition

In this section we will show that a right Bezout ring of stable range 2 is a right Hermite ring provided the transpose of any invertible matrix is invertible.

Theorem 7.9. ***(Herstein-Kaplansky)*** *A semiprimitive ring R is commutative if and only if for all $a, b \in R$ the commutator $[a, b]$ is central in R.*

Theorem 7.10. *[29] For any ring R the following statements are equivalent:*

- $\overline{R} = R/J(R)$ *is commutative;*
- *for all* n : *if* $A \in GL_n(R)$ *then* $A^T \in GL_n(R)$;
- *for any* $a,b,c \in R$: $\begin{pmatrix} 1 & a \\ -b & c \end{pmatrix} \in GL_2(R) \Rightarrow \begin{pmatrix} 1 & -b \\ a & c \end{pmatrix} \in GL_2(R)$.

Proof. (1) $\Rightarrow$ (2). Suppose that $\overline{R} = R/J(R)$ is commutative. By the definition of Jacobson radical for any ring S the element $x \in S$ is invertible if and only if $x + J(R)$ is invertible in $S/J(S)$. Hence if $A \in GL_n(R)$ then $\overline{A} \in GL_n(\overline{R})$ and the commutativity of $\overline{R}$ implies that $\overline{A^T} \in GL_n(\overline{R})$. The latter is equivalent to $A^T \in GL_n(R)$.

Since the implication (2) $\Rightarrow$ (3) is straightforward, to finish the proof it remains to show that (3) $\Rightarrow$ (2). Suppose the condition (3) holds. By Lemma 7.3 it is equivalent to

$$c + ba \in U(R) \Rightarrow c + ab \in U(R),$$

and by the above remark the same is true for $\overline{R}$. In the view of Herstein-Kaplansky theorem it suffices to show that for any $\overline{a}, \overline{b} \in \overline{R}$ the commutator $[\overline{a}, \overline{b}]$ is central in $\overline{R}$. Taking $\overline{c} = \overline{1 - ba}$ we obtain that $[\overline{a}, \overline{b}] = \overline{w} - \overline{1}$ for some $w \in U(R)$. Therefore it suffices to show that $\overline{w}$ is central in $\overline{R}$, i.e. for any $r \in R$:

$$\overline{wr} = \overline{rw} \Leftrightarrow wr - rw \in J(R) \Leftrightarrow 1 + d(wr - rw) \in U(R),$$

for every $d \in R$. If we denote by $x = 1 - drw$ we have the chain of inclusions:

$$1 = 1 - drw + drw = x + d(rw) \in U(R) \Rightarrow x + (rw)d \in U(R) \Rightarrow x + wdr \in U(R) \Rightarrow$$

$$\Rightarrow w^{-1}x + dr \in U(R) \Rightarrow w^{-1}x + rd \in U(R) \Rightarrow x + wrd \in U(R) \Rightarrow x + dwr \in U(R)$$

So $1 + d(wr - rw) \in U(R)$ and $R/J(R)$.

Theorem 7.11. *Any right Bezout ring R of stable range 2 such that transpose of any invertible matrix is invertible is a right Hermite ring.*

Proof. By [29] the transpose of any invertible matrix is again invertible if and only if $R/J(R)$ is a commutative ring. In case when $J(R) = 0$ the statement of theorem follows from [81].

So, suppose that $J(R) \neq 0$. Since $\text{st.r.}(R) = \text{st.r.}(R/J(R))$ then $\overline{R} = R/J(R)$ is a commutative Bezout ring of stable range 2 hence an Hermite ring by Theorem 3.2.

Let's prove that any unimodular row over a ring R is completable to invertible matrix. So suppose that $aR+bR=R$. Then $\overline{a}\overline{R}+\overline{b}\overline{R}=\overline{R}$ and unimodular row $(\overline{a},\overline{b})$ is completable to invertible matrix

$$\overline{A}=\begin{pmatrix}\overline{a}\,\overline{b}\\ \overline{u}\,\overline{v}\end{pmatrix}$$

as R is an Hermite ring R. Consequently there is a matrix

$$\overline{C}=\begin{pmatrix}\overline{c}\,\overline{x}\\ \overline{d}\,\overline{y}\end{pmatrix}$$

such that $\overline{AC}=\overline{CA}=\overline{I}=\mathrm{diag}\overline{1},\overline{1}$. Writing out the latter equalities elementwise we obtain

$$ac+bd=1+j_1,\ \ ax+by=j_2,\ \ uc+vd=j_3,\ \ ux+vy=1+j_4$$

for some elements $j_1,j_2,j_3,j_4\in J(R)$. Taking

$$A=\begin{pmatrix}a\,b\\ u\,v\end{pmatrix},C=\begin{pmatrix}c\,x\\ d\,y\end{pmatrix}$$

we will get

$$AC=\begin{pmatrix}1+j_1 & j_2\\ j_3 & 1+j_4\end{pmatrix}=\begin{pmatrix}1\,0\\ 0\,1\end{pmatrix},$$

where $u_1=1+j_1,\ u_4=1+j_4\in U(R)$ since $j_1,j_4\in J(R)$.

We are going to show that there is a matrix W such that $BW=WB=I$, where I is an identity matrix and $B=AC$.

Considering elements $j_2'=j_2u_4^{-1},\ j_3'=j_3u_1^{-1}\in J(R)$ we obtain

$$\begin{pmatrix}u_1\ j_2\\ j_3\ u_4\end{pmatrix}\begin{pmatrix}u_1^{-1} & 0\\ 0 & u_4^{-1}\end{pmatrix}=\begin{pmatrix}1 & j_2'\\ j_3' & 1\end{pmatrix}.$$

Using elementary transformations we obtain

$$\begin{pmatrix}1 & j_2'\\ j_3' & 1\end{pmatrix}\begin{pmatrix}1 & -j_2'\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0\\ j_3' & 1+j\end{pmatrix}$$

for some $j\in J(R)$ and

$$\begin{pmatrix}1 & 0\\ j_3' & 1+j\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & (1+j)^{-1}\end{pmatrix}=\begin{pmatrix}1 & 0\\ j_3' & 1\end{pmatrix},\ \begin{pmatrix}1 & 0\\ j_3' & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -j_3' & 1\end{pmatrix}=\begin{pmatrix}1\,0\\ 0\,1\end{pmatrix}.$$

As a result we have proved that there is a matrix B_1 such that $BB_1 = I$. In similar manner we can find a matrix B_2 such that $B_2B = I$.

Therefore $B = AC$ is an invertible matrix and $ACB^{-1} = I$. Considering similar steps for matrix $D = CA$ we obtain that D is invertible and $D^{-1}CA = I$. So, A is both right and left invertible and any unimodular row over R is completable to invertible matrix.

Now suppose that $a, b \in R$ are arbitrary. Since R is a right Bezout ring then there are $d, u, v, a_0, b_0, c_0 \in R$ such that

$$aR + bR = dR,\ a = da_0,\ b = db_0,\ au + bv = d,\ a_0u + b_0v + c_0 = 1,\ dc_0 = 0.$$

As $a_0R + b_0R + c_0R = R$ and st.r.$(R) = 2$ there are $x, y \in R$ such that

$$(a_0 + c_0x)R + (b_0 + c_0y)R = R.$$

As it was proved there is an invertible matrix

$$P = \begin{pmatrix} a_0 + c_0x & b_0 + c_0x \\ * & * \end{pmatrix}.$$

Clearly, $(d, 0)P = (a, b)$ and then

$$(a, b)P^{-1} = (d, 0),$$

i.e. R is a right Hermite ring. □

7.5 Right Bezout rings and unimodular rows

Menal and Moncasi [44] described von Neumann regular rings of finite stable range such that for any row x of length n there exists an unimodular column y of length n such that $xyx = x$. In the current section we will prove the similar description for the case of right Bezout rings of finite stable range n.

Proposition 7.4. *A ring R is a right Bezout ring if and only if for any $a, b \in R$ there exists $d \in R$ and unimodular row (a_0, b_0, c) such that $(a, b, 0) = d(a_0, b_0, c)$.*

Proof. Let R be a right Bezout ring. Then for elements $a, b \in R$ one can find elements $a_0, b_0, d, u, v \in R$ such that

$$au + bv = d,\ a = da_0,\ b = db_0.$$

If we denote by $c = 1 - a_0 u - b_0 v$ then a row (a_0, b_0, c) is right unimodular and $dc = 0$.

Conversely, suppose that $(a, b, 0) = d(a_0, b_0, c)$ and (a_0, b_0, c) is a unimodular row. Therefore one can find the elements $u, v, w \in R$ such that $a_0 u + b_0 v + cw = 1$ and also $a = da_0,\ b = db_0,\ dc = 0$. Since $dc = 0$ then $au + bv = d$ and R is a right Bezout ring. □

Similar result holds in a case of right Bezout rings of stable range n.

Theorem 7.12. *Let R be a right Bezout ring and* st.r.$(R) = n$. *Then for any row $(a_1, a_2, \ldots, a_n)$ over R there exists $d \in R$ and unimodular row $(b_1, b_2, \ldots, b_n)$ such that*

$$(a_1, a_2, \ldots, a_n) = d(b_1, b_2, \ldots, b_n).$$

Proof. As R is a right Bezout ring then there are elements $d, x_1, \ldots, x_n, u_1, \ldots, u_n \in R$ such that

$$a_1 R + \ldots + a_n R = dR,\ a_1 u_1 + \ldots + a_n u_n = d$$

and $a_1 = dx_1,\ \ldots\ , a_n = dx_n$.

Denote by $c = 1 - x_1 u_1 - \ldots - x_n u_n$. Then $dc = 0$ and

$$x_1 u_1 + \ldots + x_n u_n + c = 1.$$

Since st.r.$(R) = n$ then there are $t_1, \ldots, t_n \in R$ such that

$$(x_1 + ct_1)R + \ldots + (x_n + ct_n)R = R.$$

It is easy to check that

$$(a_1, \ldots, a_n) = d(x_1 + ct_1, \ldots, x_n + ct_n)$$

as was desired. □

The following theorem shows that the converse statement is true in case of commutative ring for $n = 2$.

Theorem 7.13. *Let R be a commutative ring such that for any row (a_1, a_2) there is $d \in R$ and unimodular row (b_1, b_2) satisfying the equality $(a_1, a_2) = d(b_1, b_2)$. Then R is a Bezout ring of stable range 2, i.e. an Hermite ring.*

Proof. Since $b_1 R + b_2 R = R$ then there are $u_1, u_2 \in R$ such that $b_1 u_1 + b_2 u_2 = 1$. Hence matrix

$$Q = \begin{pmatrix} b_1 & b_2 \\ u_1 & u_2 \end{pmatrix}$$

is invertible and

$$\begin{pmatrix} a_1 & a_2 \end{pmatrix} = \begin{pmatrix} d & 0 \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ u_1 & u_2 \end{pmatrix}.$$

If $P = Q^{-1}$ then $\begin{pmatrix} a_1 & a_2 \end{pmatrix} P = \begin{pmatrix} d & 0 \end{pmatrix}$. and R is an Hermite ring. By Theorem 3.2 it is the same as Bezout ring of stable range 2. □

Remark that the latter result can be generalized for rows of arbitrary length n over commutative rings. It is unknown remains it true or not in a noncommutative case.