Chapter 8
Rings defined by range conditions

8.1 Neat range one and stable range

Proposition 8.1. *Let R be a commutative ring of neat range 1. Then* $\mathrm{st.r.}(R) = 2$.

Proof. Suppose that $aR + bR + cR = R$ for $a,b,c \in R$. Let

$$bR + cR = \delta R, \ \ aR + \delta R = R$$

for some $\delta \in R$. Since R is a ring of neat range 1, then there exist $y,z \in R$ such that $w = a + by + cz$ and $\overline{R} = R/wR$ is clean ring. Since a,b,c are coprime then $ak + bp + cq = 1$ for some $k,p,q \in R$. As the result

$$(a + by + cz)k + b(p - yk) + c(q - zk) = 1,$$

i.e $wR + bR + cR = R$. Then $\overline{b}\overline{R} + \overline{c}\overline{R} = \overline{R}$. Since $\overline{R}$ is clean ring, then $\overline{R}$ is a ring of idempotent stable range 1 due to Proposition 2.17, i.e. there exists an idempotent $\overline{e} \in \overline{R}$ such that $(\overline{b} + \overline{ce})\overline{R} = \overline{R}$. Hence $(b + ce)R + wR = R$, where $\overline{b} = b + wR$, $\overline{c} = c + wR$, $\overline{e} = e + wR$. The letter equality implies that $ws + (b + cd)t = 1$ for some $s,t \in R$.

Then $(a + (b + cd)y + c(z - dy))s + (b + cd)t = 1$ and so $(a + c(z - dy))s + (b + cd)(t + ys) = 1$, $(a + c\alpha)R + (b + c\beta)R = 1$ for the appropriate $\alpha, \beta \in R$. Hence st.r. $= 2$. Proposition is proved. □

Since a commutative Bezout ring R is an Hermite ring if and and only if $\mathrm{st.r.}(R) = 2$ hence by the previous theorem we have the following result.

Theorem 8.1. *A commutative Bezout ring is an elementary divisor ring if and only if it is a ring of neat range 1.*

8.2 Semihereditary and von Neumann regular range one

This chapter deals mostly with the question: when is the classical ring of quotients of a commutative ring of a ring of stable range 1 and its generalization.

We introduces the concept of a ring of von Neumann regular range 1, a ring of semihereditary range 1, a ring of regular range 1, a semihereditary local ring, a regular local ring. Found relationship between introduced classes ring and famous in particular installed that a commutative indecomposable almost clean ring it is a regular local ring. A commutative ring of idempotent regular range 1 is an almost clean ring. It is shown that a commutative indecomposable almost clean Bezout rings are Hermite rings, a commutative semihereditary ring are a ring of idempotent regular range 1. A classical ring of quotients of a commutative Bezout ring $Q_{Cl}(R)$ is a regular local ring if and only if R is a commutative semihereditary local ring.

Throughout, all rings are assumed to be associative with unit and $1 \neq 0$. The set of nonzero divisors (also, called regular elements) of R will be denoted $\mathfrak{R}(R)$, the set of units is $U(R)$ and the set of idempotents $\mathfrak{B}(R)$. A classical ring of quotients of ring R is written $Q_{Cl}(R)$.

Definition 8.1. A ring R is called *indecomposable* if $\mathfrak{B}(R) = \{0,1\}$.

Definition 8.2. An element a of a ring R is called *left (right) semihereditary element* if Ra (aR) is projective $R-$module.

Proposition 8.2. *Let R is a commutative Bezout ring. If $\phi \in \mathfrak{B}(Q_{Cl}(R))$ then $\phi \in \mathfrak{B}(R)$.*

Proof. Let $\phi \in \mathfrak{B}(Q_{Cl}(R))$ and $\phi = \frac{e}{s}$, where s is a regular element of R. Let $eR + sR = \delta R$, then $e = e_0\delta$, $s = s_0\delta$ and $eu + sv = \delta$ for some elements $e_0, s_0, u, v \in R$. Since s is a regular element then δ is a regular element as the divisor of s. Since $eu + sv = \delta$ then $\delta(e_0u + s_0v - 1) = 0$. Since $\delta \neq 0$ and δ is a regular element of R, we have $e_0u + s_0v - 1 = 0$. Then $\frac{e}{s} = \frac{e_0}{s_0}$, where $e_0R + s_0R = R$. Since $\frac{e_0}{s_0} \in \mathfrak{B}(Q_{Cl}(R))$ then $e_0^2 s_0 = e_0 s_0^2$ and $s_0(e_0^2 - e_0 s_0) = 0$. Since $s_0 \neq 0$ and so s_0 is a regular element of R as the divisor of s, we have $e_0^2 = e_0 s_0$.

Since $e_0u + s_0v = 1$ we have $e_0^2 u + e_0 s_0 v = e_0$ and $s_0(e_0u + s_0v) = e_0$. Hence $\frac{e_0}{s_0} \in R$. Proposition is proved. □

Proposition 8.3. *Let R is a commutative ring and a is von Neumann regular element of R. Then $a = eu$, where $e \in \mathfrak{B}(R)$ and $u \in U(R)$*

Proof. Let $axa = a$. From here we have $axax = ax$, i.e. $e = ax \in \mathfrak{B}(R)$ and $e \in aR$. Since $axa = a$, then $ea = a$, i.e. $a \in eR$ and we have $aR = eR$.

Consider element $u = (1-e)+a$. Since $u(1-e) = 1-e$, we have $uR+eR = R$. We proved that $eR = aR$, then $uR+aR = R$. Since $ue = ((1-e)+a)e = ae = a$, then $aR \subset uR$. Obviously equality $uR+aR = R$ and inclusion $aR \subset uR$ in a commutative ring possible if $u \in U(R)$.

Then we have $ue = a$. Proposition is proved. □

Proposition 8.4. *Let R is a commutative ring and a is a semihereditary element if and only if $a = er$, where $e \in \mathfrak{B}(R)$ and $r \in \mathfrak{R}(R)$.*

Proof. Let $\phi R = \{x \mid xa = 0\}$ and $\phi \in \mathfrak{B}(R)$. Since $\phi a = 0$, we have $(1-\phi)a = 1-\phi$. Let $r = a-\phi$ and $rx = 0$.

Since $ax = \phi x$ and $(1-\phi)a = a$ we have $(1-\phi)ax = ax$ and $(1-\phi)\phi x = 0$. Then $ax = 0$ and $\phi x = 0$. Since $ax = 0$, we have $x \in \phi R$, i.e. $x = x\phi$. Since $x\phi = 0$ we have $x = 0$. Then we have r is regular element of R. Since

$$r(1-\phi) = a(1-\phi) - \phi(1-\phi) = a(1-\phi) = a,$$

i.e. $a = r(1-\phi)$. Put $1-\phi = e$ we have $a = re$ where $e \in \mathfrak{B}(R)$ and $r \in \mathfrak{R}(R)$. Obviously $\{x|x(re) = 0\} = (1-e)R$. Proposition is proved. □

Definition 8.3. A ring R is said to have *a von Neumann regular range 1* if for any $a,b \in R$ such that $aR+bR = R$ there exists $y \in R$ such that $a+by$ is (von Neumann) regular element of R.

Obviously example of ring von Neumann regular range 1 is a ring of stable range 1. Moreover, we have the next result.

Proposition 8.5. *A commutative ring of von Neumann regular range 1 is a ring of stable range 1.*

Proof. Let R is a ring of von Neumann regular range 1 and $aR+bR = R$. Then exists an element $y \in R$ such that $a+by = r$ is von Neumann regular element of R. By the Proposition 8.3 we have $a+by = r = ek$, where $e \in \mathfrak{B}(R)$ and $k \in U(R)$.

Note, since $aR+bR = R$ we have $eR+bR = R$. Then $eu+bv = 1$ for some elements $u,v \in R$. Since $1-e = (1-e)eu+(1-e)bv$, we have $1-e = (1-e)bv$, and $e+b(1-e)v = 1$. Then $ek+b(1-e)kv = k$.

So we have $a+bs = k$ for some element $s \in R$, i.e. $(a+bs)R = R$.By we have that R is a ring of stable range 1. Proposition is proved. □

Then we have a next result.

Theorem 8.2. *For a commutative ring the following conditions are equivalent:*

1. R is a ring of stable range 1;

2. R is a ring of von Neumann regular range 1.

Definition 8.4. A ring R is said to have *a semihereditary range 1* if for any $a,b \in R$ such that $aR+bR=R$ there exists $y \in R$ such that $a+by$ is a semihereditary right element of R.

Obviously example of a ring of semihereditary range 1 is a ring of stable range 1 and a commutative semihereditary ring.

Special place in a class ring a semihereditary range 1 take up a semihereditary local ring.

Definition 8.5. A commutative ring R is *a semihereditary local ring* if for any $a,b \in R$ such that $aR+bR=R$ we have a or b is a semihereditary element of R.

Obviously example of a semihereditary local ring is von Neumann regular local ring and a semihereditary ring, i.e. a commutative domain (that is not a field) is a semihereditary local ring which is not von Neumann regular local ring.

Proposition 8.6. *A commutative semihereditary local ring is a ring of semihereditary range 1.*

Proof. Let R is a commutative semihereditary local ring and $aR+bR=R$. If a is a semihereditary element. We have representation $a+b0$ is a semihereditary element. It a is not a semihereditary, by with condition $aR+(a+b)R=R$ we have $a+b1$ is a semihereditary element. Proposition is proved. □

A ring $\mathbb{Z}_{36}$ is not a semihereditary local ring, but $\mathbb{Z}_{36}$ is a ring of semihereditary range 1.

Definition 8.6. A ring R is said to have *a von Neumann regular range 1* if for any $a,b \in R$ such that $aR+bR=R$ there exist $y \in R$ such that $a+by$ is a regular element of R.

Theorem 8.3. *For a commutative ring R the following condition are equivalent.*

1. R is a ring of von Neumann regular range 1.

2. R is a ring of semihereditary range 1.

Proof. It is evident that a von Neumann regular element is a semihereditary element, so if R is a ring of von Neumann regular range 1 then R is a ring of semihereditary range 1.

Let R is a ring of semihereditary range 1 and $aR+bR=R$. Then there exist $y \in R$ such that $a+by$ is a regular element of R. By Proposition 8.4 we have $e \in \mathfrak{B}(R)$, $r \in \mathfrak{R}(R)$. Since $aR+bR=R$ we have $eR+bR=R$. Then $eu+bv=1$ for some elements $u,v \in R$. Since $1-e=(1-e)eu+(1-e)bv$ we have $e+b(1-e)v=1$ and $er+br(1-e)v=r$. Since $a+by=er$ and $er+br(1-e)v=r$ we have $a+bs=r$ for some element $s \in R$. Then R is a ring of von Neumann regular range 1. Proposition is proved. □

Proposition 8.7. *A classical ring of quotients $Q_{Cl}(R)$ of a commutative Bezout ring R of von Neumann regular range 1 is a ring of stable range 1.*

Proof. Let
$$\frac{a}{s}Q_{Cl}(R)+\frac{b}{s}Q_{Cl}(R)=Q_{Cl}(R).$$
Then $au+bv=t$ where $u,v \in R$ and $t \in \mathfrak{R}(R)$. Since R is a commutative Bezout ring, we have $aR+bR=dR$ for some element $d \in R$. Then $a=a_0d$, $b=b_0d$ and $ax+by=d$ for some elements $a_0,b_0,x,y \in R$. Since $au+bv=t$, we have $d(a_0u+b_0v)=t$. Then d is a regular element as the divisor of a regular element t.

Since $d(a_0x+b_0y-1)=0$ and $d \neq 0$ we have $a_0x+b_0y-1=0$ i.e. $a_0R+b_0R=R$. Since R is a ring of von Neumann regular range 1 we have $a_0+b_0k=r$ regular element of R for some element $k \in R$. Then $a+bk=rd \in \mathfrak{R}(R)$. So we have $\frac{a}{s}+\frac{b}{s}k=\frac{rd}{s}$.

Since $\frac{rd}{s} \in U(Q_{Cl}(R))$ we have $(\frac{a}{s}+\frac{b}{s}k)Q_{Cl}(R)=Q_{Cl}(R)$ i.e $Q_{Cl}(R)$ is a ring of stable range 1. Proposition is proved. □

Here are some examples of a rings of von Neumann regular range 1.

Definition 8.7. A commutative ring R is *a regular local ring* if for any $a \in R$ we have a or $1-a$ is a regular element.

Proposition 8.8. *A commutative regular local Bezout ring is a ring of stable range 2.*

Proof. Let R is a regular local Bezout ring. Let a,b is any nonzero elements of R. Since R is a commutative Bezout ring, we have $aR+bR=dR$. Then we have $au+bv=d$, $a=a_0d$, $b=b_0d$ for some elements $a_0,b_0,u,v \in R$. Since $d(a_0u+b_0v-1)=0$ by definition of a ring R we have that a_0u+b_0v or a_0u+b_0v-1 is a regular element of R. If a_0u+b_0v-1 is a regular element by $d(a_0u+b_0v-1)=0$ we have $d=0$, i.e. $a=b=0$ and it is impossible. Let $a_0u+b_0v=r$ is a regular element of R.

Let $a_0R+b_0R=\delta R$. If $\delta\notin U(R)$ we have $a_0x+b_0y=\delta$, $a_0=\delta a_1$, $b_0=\delta b_1$ for some elements $a_1,b_1,x,y\in R$. From here we have $\delta(a_1u+b_1v)=a_0u+b_0v=r$. Since $r\in\mathfrak{R}(R)$, we have $\delta\in\mathfrak{R}(R)$.

From here we have $\delta(a_1x+b_1y-1)=0$ and since $\delta\neq 0$ we have $a_1x+b_1y-1=0$ i.e. $a_1R+b_1R=R$. So we have $a=d\delta a_1$, $b=d\delta b_1$, $a_1R+b_1R=R$. By [**?**] R is a Hermite ring and by Theorem 3.2 we have that R is a ring of stable range 2. Proposition is proved. □

In a class rings of von Neumann regular range 1 allocate of a class of ring of idempotent von Neumann regular range 1.

Proposition 8.9. *A ring R is said to be a ring of idempotent von Neumann regular range 1 if for any element $a,b\in R$ such that $aR+bR=R$ there exists idempotent $e\in\mathfrak{B}(R)$ and regular element $r\in\mathfrak{R}(R)$ such that $a+be=r$.*

Obvious example of a ring of idempotent von Neumann regular range 1 is a ring of idempotent stable range 1 i.e a commutative clean ring.

Proposition 8.10. *A commutative regular local ring is a ring of idempotent von Neumann regular range 1.*

Proof. Let R is a regular local ring and $aR+bR=R$. If a is a regular element we have a representation $a+b0=a$. If a is not a regular element since $aR+(a+b)R=R$ we have $a+b1$ is a regular element. Proposition is proved. □

Theorem 8.4. *A commutative semihereditary ring is a ring of idempotents von Neumann regular range 1.*

Proof. Let R is a commutative semihereditary ring and $aR+bR=R$. By [**?**] and Proposition 8.4 we have $a=er$ where e is idempotent and r is a regular element. Note if $e=1$, we have a is a regular element and $a+b\cdot 0$ is necessary representation. Of $e\neq 1$, let $s=a+b(1-e)$. Show that s is a regular element of R. Let $sx=0$, then $ax=-b(1-e)x$. Since $a=er$, we have

$$erx=(1-e)(-b)x.$$

So we have $e\cdot erx=e(1-e)(-b)=0$. Since $erx=exr=0$ and r is regular nonzero element, we have $b(1-e)x=0$ we have $bx=bex=0$. Hence we have

$ax = 0$ and $bx = 0$. Since $aR + bR = R$ we have $au + bv = 1$ for some elements $u, v \in R$. Then $x = axu + bxv = 0$ and $s = a + b(1 - e)$ is a regular element. So we have R is a ring of idempotent von Neumann regular range 1. Theorem is proved. □

Proposition 8.11. *A commutative ring of idempotent von Neumann regular range 1 is an almost clean ring.*

Proof. Let R is a ring of idempotent von Neumann regular range 1 and $a \in R$ is any nonzero element $a \in R$. Then $aR + (-1)R = R$ and $a - e = r$ where e is idempotent and r – regular element of R. Proposition is proved. □

Question 8.1. Is every a commutative almost clean ring a ring of idempotent von Neumann regular range 1?

Proposition 8.12. *For a commutative ring R the following conditions we equivalent:*

1. R – indecomposable almost clean ring;

2. R – regular local ring.

Proof. Let R is an indecomposable almost clean ring. Since 0 and 1 is all idempotent of R, we have for any a then a or $1 - a$ is a regular element of R.

Let R is a regular local ring. Since for each idempotent $e \in R$ we have, that e and $1 - e$ is an idempotent we have that R is indecomposable ring. By Proposition 8.10 we have that R is a ring of idempotent von Neumann regular range 1 and by Proposition 8.11 R is an almost clean ring. Proposition is proved. □

By Theorem 3.2 and Proposition 8.8 we have a following result.

Theorem 8.5. *A commutative indecomposable almost clean Bezout ring is an Hermite ring.*

Proposition 8.13. *A commutative semihereditary local ring is a ring of idempotent von Neumann regular range 1.*

Proof. Let R is a commutative semihereditary local ring and $aR + bR = R$. If a is semihereditary element we have a representation $a = er$, where e is idempotent and r is a regular element. Then we have $a + b(1 - e)$ is a regular element by the proof of Theorem 8.4. If a is not a semihereditary element then with equality $aR + (a + b)R = R$ we have $a + b = er$ is a semihereditary element i.e. $e^2 = e$ and $r \in \mathfrak{R}(R)$.

Since $(a + b)R + (-b)R = R$ we have $a + b - b(1 - e) = a + be = s$ we have a necessary representation Proposition is proved. □

Theorem 8.6. *Let R is a commutative Bezout ring. Then $Q_{Cl}(R)$ is von Neumann regular local ring if and only if R is a semihereditary local ring.*

Proof. Let $aR+bR=R$ then $\frac{a}{1}Q_{Cl}(R)+\frac{b}{1}Q_{Cl}(R)=Q_{Cl}(R)$. Since $Q_{Cl}(R)$ is von Neumann regular local ring we have that $\frac{a}{1}$ or $\frac{b}{1}$ is von Neumann regular element. If $\frac{a}{1}$ is von Neumann regular element then by Proposition 8.3 we have $\frac{a}{1}=eu$ where $e^2=e\in Q_{Cl}(R)$ and $u\in U(Q_{Cl}(R))$. By Proposition 8.2 we have $e\in R$. Then we have $a=er$, where r is a regular element of R. The case when $\frac{b}{1}$ is von Neumann is similar.

Let R is a semihereditary local ring and

$$\frac{a}{s}Q_{Cl}(R)+\frac{b}{s}Q_{Cl}(R)=Q_{Cl}(R)$$

and $\frac{a}{s}\neq 0$ or $\frac{b}{s}\neq 0$. Then $au+bv=t$ for some elements $u,v\in R$ and t-regular element R. Since R is a commutative Bezout ring, then $aR+bR=dR$. Let $a=a_0d$, $b=b_0d$ and $ax+by=d$ for some elements $a_0,b_0,x,y\in R$. With equality $au+bv=t$ we have $d(a_0u+b_0v)=t$. Then d is a regular element as a divisor of t. With equality $ax+by=d$ we have $d(a_0x+b_0y-1)=0$. Since $d\neq 0$ we d is a regular element we have $a_0x+b_0y=1$. Hence $a_0R+b_0R=R$ we have a_0 or b_0 is a semihereditary element.

If a_0 is a semihereditary element by Proposition 8.4 we have $a_0=er$, where $e^2=e$ and r is a regular element of R. Since $a=a_0d=e(rd)$, we have $\frac{a}{s}=e\frac{rd}{s}$. Since $e^2=e$ and $\frac{rd}{s}\in U(Q_{Cl}(R))$ we have that $\frac{a}{s}$ is von Neumann regular element. If b_0 is von Neumann regular we have similar proof. Then $Q_{Cl}(R))$ is von Neumann regular local ring. Theorem is proved. □

Definition 8.8. A commutative ring R is said to be *additely regular* if for each $a\in R$ and each regular element $b\in R$ there exist en element $u\in R$ such that $a+ub$ is regular in R [26].

Proposition 8.14. *A commutative Bezout ring of von Neumann regular range 1 is additively regular.*

Proof. Let R is a commutative Bezout ring of von Neumann regular range 1 and a is any element R and b is any regular element of R. Since R is a commutative Bezout ring we have $aR+bR=dR$ and where $au+bv=d$, $a=a_0d$, $b=b_0d$ for some element $u,v,a_0,b_0\in R$. Since b is a regular element of R, we have d is a regular element of R, since d is divisor of b.

Since $au+bv=d$ we have $d(a_0u+b_0v-1)=0$. Hence $d\neq 0$ we have $a_0u+b_0v-1=0$ i. e. $a_0R+b_0R=R$. So R is a ring of von Neumann regular range 1 we have $a_0 + b_0 t = r$ is regular element for some $t \in R$. Then $a + bt = rd$ – regular ring i.e. R is an additively regular ring. Proposition is proved. □

8.3 Additive range one

8.4 Dyadic range one

One of the main sources of almost all researches in the present paper is a rather problem of a full desoripton of the elementary divisor rings. The notion of an elementary divisor ring was introduced by Kaplansky [**?**]. Recall that a matrix over an associative ring has a canonical diagonal reduction if can be reduced to a diagonal form by left and right multiplication by some invertible matrices of the corresponding sizes and so that each diagonal element is a full divisor of the following one. If any matrix over the ring has a canonical diagonal reduction then such ring is called an elementary divisor ring **[ ]**. In the same paper, Kaplansky showed that on an elementary divisor ring any finitely presented module can be decomposed into a direct sum of cyclic modules. In the case of commutative rings the reverse statement is proved namely: if any finitely presented module over a ring can be decomposed into a direct sum of cyclic modules then this ring is an elementary divisor ring [43]. This result is a partial solution to the problem of Warfield [**?**].

There are a lot of researches that deal with the matrix diagonalization in different cases (the most comprehensive history of these researches ban found in [**?**]).

Specific role in modern research on elementary divisor rings plays a K-theoretical invariant such us the stable range that was established in 1960 by Bass [**?**]. One of the most fruitful aspects of Bass's studies was the following fact: a lot of answers to the problems of the linear algebra over rings becomes simples if we increase the dimension of the consider object (the rank of the projective module, the size of the matrix etc.) and, furthermore, the answer is independent on the choice of the base ring for rather big dimensions' values, as well as is independent on the dimension of the current object – in functions only in terms of the ”geometry” of a given module. Moreover, it have been discovered that in the commutative ring cases there exist structural theorems on the these objects starting from some small values of a stable range (for example, 1 or 2) which depends only the consider problem, but does not depend on the dimension or structure of the base ring.

For example, an important role in studying of the elementary divisor rings is played by the Hermitian rings. A ring is called right (left) Hermitian if all 1×2 (2×1) matrix over this ring have diagonal reduction over this ring. An Hermitian ring is a ring which is both right and left Hermitian [**?**].

Definition 8.9. Let R be a commutative Bezout domain. We say that R is a ring of Gelfand range 1 if for any $a, b \in R$ such that $aR + bR = R$ there exist an element $\lambda \in R$ such that a factor-ring $R/(a + b\lambda)R$ is a Gelfand ring [**?**].

The main result is a next theorem.

Theorem 8.7. *[95] A commutative Bezout domain is an elementary divisor ring if and only if it is a ring of Gelfand range 1.*

This result give a solution of the problem of elementary divisor rings for different classes of a commutative Bezout domains in partical for a PM* ring local Gelfand domains an so on [**?**].

In this paper based on the concept of a ring dyadic range 1 we similarly describe commutative elementary divisor ring.

Let R be assotiative ring with unit and $1 \neq 0$.

Definition 8.10. Let $a, b \in R$ and $aR + bR = R$. Will say that pair (a, b) *has a right diadem* (or (a, b) is *a right dyadic pair*) if there are an element $\lambda \in R$ such that for element $a + b\lambda$ and any elements $c, d \in R$ such that $(a + b\lambda)R + cR + dR = R$ there are an element $\mu \in R$ such that $(a + b\lambda)R + (c + d\mu)R = R$. Call element $a + b\lambda$ a right diadem of pair (a, b). A left diadem and left diadic pair can be introduced analogously. A right and left diadem we will simply call a *diadem*.

Example 8.1. An obvious example of diadic pair (that may also be called a *trivial diadic pair*) is (a, u), where u is an invertible element of a ring R and a is any element of R. Here $u + a0$ and $a + (-au^{-1} + 1)u$ are right diadems of pair (a, u).

To obtain a nontrivial example take a pair $(a, a + u)$, where $a \in R$ and u is an invertible element of R. Therefore $a + (a + u) - 1$, $(a + u) + a(-1)$ are a right diadems of pair $(a, a + u)$.

Definition 8.11. If for right diadic pair (a, b) exist element $\lambda \in R$ such that $a + b\lambda$ is invertible element, then diadem $a + b\lambda$ will call *trivial diadem*.

Example 8.2. It is well known that every semiperfect ring, unit-regular ring and strongly π-regular ring has stable range 1. Meanwhile, every left (right) quasi-duo exchange ring has stable range 1 and every exchange ring of bounded index of nilpotency has stable range 1.

Remark 8.1. Chen [9] has shown that an abelian ring is clean if and only if it has idempotent stable range 1. Note that in a ring of stable range 1 any right dyadic pair (a,b) has a trivial diadem.

Definition 8.12. We say that a ring R is a ring of *a right dyadic range 1* if for any elements a,b the equality $aR+bR=R$ implies that a pair (a,b) has a right diadem. Similarly, we define a ring of a left dyadic range 1. A ring of right and left dyadic range 1 is called *a ring of dyadic range 1.*

Example 8.3. Any ring of stable range 1 is a ring dyadic range 1.

Moreover, we have a following result.

Theorem 8.8. *Bezout ring of right dyadic range 1 is a ring of stable range 2.*

Proof. Let $aR+bR+cR=R$. Since R is a right Bezout ring, then $bR+cR=dR$ and $aR+dR=R$. Let $\nu=a+d\lambda$ is a right diadem of pair (a,d). From here $\nu=a+bx+cy$ for some elements $x,y\in R$. Note, that $\nu R+bR+cR=R$. Since $aR+bR+cR=R$. According of definition a ring R of right dyadic range 1, we have $\nu R+(b+c\mu)R=R$ for some element $\mu\in R$. Then $\nu s+(b+c\mu)t=1$ for some elements $s,t\in R$. Note, that $Rs+Rt=R$. Since $\nu=a+bx+cy$ then $(a+bx+cy)s+(b+c\mu)t=1$ and $as+b(xs+t)+c(ys+\mu t)=1$. Since $Rs+Rt=R$, then $Rs+R(xs+t)=R$. Really if $Rs+R(xs+t)=Rh$ where h is not invertible element of R then $s=s_0h$, $xs+t=\lambda h$ for some elements so $\lambda\in R$. Then $\lambda h=xs+t-xs_0h+t$ and $t=(\lambda-xs_0)h$ it is impossible, since $Rs+Rt=R$. So $Rs+R(xs+t)=R$, from here $us+\mu(xs+t)=ys+\mu t$ for some elements $u,v\in R$. Since $as+b(xs+t)+c(ys+\mu t)=1$ and $us+\nu(xs+t)=ys+\mu t$ we had $as+b(xs+t)+cus+c\nu(xs+t)=1$ that $(a+cu)s+(b+c\nu)(xs+t)=1$ and $(a+cu)R+(b+c\nu)R=R$. Thus R is a ring of stable range 2. Theorem is proved.

Proposition 8.15. *A ring R is a ring of right dyadic range 1if and only if $R/J(R)$ is a ring of right dyadic range 1, where $J(R)$ – Jacobson radical of R.*

Proof. Let $\overline{R}=R/J(R)$ and $\overline{a}\overline{R}+\overline{b}\overline{R}=\overline{R}$, where $\overline{a}=a+J(R)$, $\overline{b}=b+J(R)$. Since, $\overline{a}\overline{R}+\overline{b}\overline{R}=\overline{R}$, then $aR+bR=R$. Since R is a ring of right dyadic range 1 then exists element $\lambda\in R$ such that $a+b\lambda$ is right diadem pair (a,b). Let $\overline{\lambda}=\lambda+J(R)$ and $(\overline{a}+\overline{b}\overline{\lambda})\overline{R}+\overline{c}\overline{R}+\overline{d}\overline{R}=\overline{R}$ where $\overline{c}=c+J(R)$, $\overline{d}=d+J(R)$. Then $(a+b\lambda)R+cR+dR=R$ and since $a+b\lambda$ is right diadem pair (a,b). We have $(a+b\lambda)R+(c+d\mu)R=R$ for some element $\mu\in R$ and $(\overline{a}+\overline{b}\overline{\lambda})\overline{R}+(\overline{c}+\overline{d}\overline{\mu})\overline{R}=\overline{R}$ thus $\overline{a}+\overline{b}\overline{\lambda}$ is a right diadem pair $(\overline{a},\overline{b})$. If $\overline{R}$ is right of right dyadic range 1, then the fact that $\overline{R}=R/J(R)$ is a ring of right diadic range 1 is obviously. Theorem is proved.

Proposition 8.16. *Let R is a ring of right diadic range 1 and $aR = bR$. Then exists diadems d_1, d_2 for some dyadic pairs of R such that $ad_1 = b$, $bd_2 = a$*

Proof. Since $aR = bR$ then $a = bs$, $b = at$ for some elements $s,t \in R$. From here $a(1-ts) = 0$ and $1-ts \in \mathrm{Ann}(a)$. Note if $\mathrm{Ann}(a) = (0)$ then $ts = 1$. By Theorem 8.8, R is a ring of stable range 2, by [9] R is finite, i.e. t,s is invertible ring and proposition is proved.

Let $\mathrm{Ann}(a) \neq (0)$. Since $1-ts \in \mathrm{Ann}(a)$, then $tR + \mathrm{Ann}(a) = R$. From here, we have $tx + \lambda = 1$ for some elements $x \in R$ and $x \in \mathrm{Ann}(a)$. Since R is a ring of right diadic range 1, let $t + \alpha\lambda = d_1$ is diadem pair (t, α). Then $at + a\alpha\lambda = ad_1$, since $a\alpha = 0$, and $at = b$, we have $b = ad_1$. Which have similar $bd_2 = a$ for some right diadem.Proposition is proved.

Moreover, for a class right quasi morphic ring we have inverse statement.

Proposition 8.17. *Let R is a right quasi-morphic ring in which the condition $aR = bR$ followed that $a = bd_1$, $b = ad_2$ for some diadems $d_1, d_2 \in R$. Then R is a ring of right dyadic range 1.*

Proof. Let $xR + yR = 1$, then $xz - 1 \in yR$ for some element $z \in R$. Let $yR = \mathrm{Ann}(\alpha)$ and $\alpha xR = \mathrm{Ann}(\beta)$ for some elements $\alpha, \beta \in R$. The following elements are since R is right quasi morphic ring. Note that for any element $r \in R$ we have $(\beta\alpha)xr = \beta(\alpha x)r = 0$ and $(\beta\alpha)yr = \beta(\alpha y)r = 0$ since $\alpha(yr) = 0$ for any element $r \in R$. So $xR \subseteq \mathrm{Ann}(\beta\alpha)$ and $yR \subseteq \mathrm{Ann}(\beta\alpha)$. Since $xR + yR = R$, we have $1 \in \mathrm{Ann}(\beta\alpha)$, i.e. $\beta\alpha = 0$. So $\alpha \in \mathrm{Ann}(\beta)$, i.e. $\alpha R \subseteq \mathrm{Ann}(\beta)$. Also we have $\mathrm{Ann}(\beta) = \alpha xR \subset \alpha R$.

Therefore we have $\mathrm{Ann}(\beta) = \alpha xR = aR$. Under the conditions imposed on R, we have with conditions $dxR = \alpha R$ follows $\alpha x = \alpha d$ for some right diadem $d \in R$. So $d(x-d) = 0$ i.e. $x - d \in \mathrm{Ann}(a) = yR$. From here we have $x + y\lambda = d$ for some elements $\alpha \in R$ that R is ring of right dyadic range 1. Proposition is proved.

For next results will be useful a next result.

Proposition 8.18. *Let R commutative ring and pair (a,b) is dyadic pair. Element $a + b\lambda$ is diadem if and only if factor-ring $R/(a+b\lambda)R$ is a ring of stable range 1.*

Proof. Let $a + b\lambda$ is diadem. Denoted $\overline{R} = R/(a+b\lambda)R$ and let $\overline{c}\overline{R} + \overline{d}\overline{R} = \overline{R}$, where $\overline{c} = c + (a+b\lambda)R$, $\overline{d} = d + (a+b\lambda)R$. Since $\overline{c}\overline{R} + \overline{d}\overline{R} = \overline{R}$ then $(a+b\lambda)R + cR + dR = R$. Since $a + b\lambda$ is diadem, then exist $\mu \in R$ such that $(a+b\lambda)R + (c + d\mu)R = R$. From here $(\overline{c} + \overline{d}\overline{\mu})\overline{R} = \overline{R}$ where $\overline{\mu} = \mu + (a+b\lambda)R$. Thus proved that stable range of $\overline{R} = R/(a+b\lambda)R$ is equal 1. Sufficiency is obvious. Proposition is proved.

Proposition 8.19. *Let R is a commutative ring and let $c \in R \setminus \{0\}$. If for any $a, b \in R$ such that $aR + bR + cR = R$ there exist $r, s \in R$ such that $c = rs$ and $rR + sR = R$, $rR + aR = R$, $sR + bR = R$ then R/cR is an exchange ring.*

Proof. Denote $\overline{R} = R/cR$. Since $aR + bR + cR = R$, then $\overline{b}\overline{R} + \overline{a}\overline{R} = \overline{R}$ where $\overline{b} = b + cR$, $\overline{a} = a + cR$. Denote $\overline{R} = r + cR$, $\overline{s} = s + cR$. Since $rR + sR = R$, one has $ru + sv = 1$. We have $\overline{R}^2\overline{u} = \overline{R}\overline{u}$, $\overline{s}^2\overline{v} = \overline{s}\overline{v}$ where $\overline{u} = u + cR$, $\overline{v} = v + cR$. Let $\overline{s}\overline{v} = \overline{e}$, obviously $\overline{e}^2 = \overline{e}$ and $\overline{1} - \overline{e} = \overline{R}\overline{u}$. Since $rR + aR = R$, we obtain $r\alpha + a\beta = 1$ for some elements $\alpha, \beta \in R$ then $rsv\alpha + asu\beta = sv$ and then $\overline{a}\overline{e}\overline{\beta} = \overline{e}$ where $\overline{\beta} = \beta + cR$. Similarly $\overline{b}\overline{x}(\overline{1} - \overline{e}) = \overline{1} - \overline{e}$ for some element $\overline{x} \in \overline{R}$. We proved that if $\overline{a}\overline{R} + \overline{b}\overline{R} = \overline{R}$ then there exist an idempotent $\overline{e}$ such that $\overline{e} \in \overline{a}\overline{R}$ and $\overline{1} - \overline{e} \in \overline{b}\overline{R}$ i. e. R is an exchange ring. Proposition is proved.

Proposition 8.20. *Let R is a commutative Bezout ring of diadic range 1. Then for any divisor α of diadem $a + b\lambda$ and elements $c, d \in R$ such that $\alpha R + cR + dR = R$ exist element $\mu \in R$ such that $\alpha R + (c + d\mu)R = R$.*

Proof. Let $a + b\lambda = \alpha\beta$ and $cR + dR = kR$. Since R is a ring of diadic range one, then by Theorem **??** and Theorem 8.8 we have, that $c = kc_1$, $d = kd_1$ for some element $c_1, d_1 \in R$ such that $c_1R + d_1R = R$.

Since $(a + b\lambda)R + c_1R + d_1R = R$ there exist element $\mu \in R$ such that $(a + b\lambda)R + (c_1 + d_1\mu)R = R$. From here $\alpha R + (c_1 + d_1\mu)R = R$. Since $\alpha R + cR + dR = R$ and $cR + dR = kR$ we have $\alpha R + kR = R$. Equality $\alpha R + (c_1 + d_1\mu)R = R$ provides $R = \alpha R + k(c_1 + d_1\mu)R = \alpha R + (c + d\mu)R$. Proposition is proved. □

Theorem 8.9. *A commutative Bezout ring is an elementary divisor ring if and and only if it is a ring of dyadic range 1.*

Proof. Let R is a commutative Bezout ring of dyadic range 1. By Theorem **??** and Theorem 8.8 we have that R is a Hermite ring. Suppose that $A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in M_2(R)$ with $aR + bR + cR = R$. By [**?**] it is suffices ti check that A admits an elementary reduction.

Since R is a ring of dyadic range 1, there exist some elements $x, y \in R$ such that $b + ax + cy = w$ is diadem. Then $\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ w & c \end{pmatrix}$. Obviously $aR + wR + cR = R$. Since R is an Hermite ring, there exists invertible matrix $Q \in GL_2(R)$ such that $(w, c)Q = (\alpha, 0)$ for some element $\alpha \in R$. Obviously α is a divisor of diadem w. Let $\begin{pmatrix} a & 0 \\ w & c \end{pmatrix} Q = \begin{pmatrix} a' & c' \\ 0 & \alpha \end{pmatrix}$ with some elements $a', c' \in R$. It is easily seen that $\alpha R + a'R + c'R = R$. By Proposition 8.20 there exists element $\mu \in R$ such that $\alpha R + (c' + a'\mu)R = R$. Then $\begin{pmatrix} a' & c' \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} \mu & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} c' + a'\mu & a' \\ \alpha & 0 \end{pmatrix}$. Since R is an Hermite ring and $\alpha R + (c' + a'\mu)R = R$ there exist invertible matrix $P \in GL_2(R)$ such that $P \begin{pmatrix} c' + a'\mu & a' \\ \alpha & 0 \end{pmatrix} = \begin{pmatrix} 1 & * \\ * & * \end{pmatrix} = B$. Obviously matrix B admits a diagonal reduction. Therefore $A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ admits a diagonal reduction and so R is an elementary divisor ring.

Let R is an elementary divisor ring. By [75] for some element $a, b \in R$ such that $aR + bR = R$ exists element $\lambda \in R$ such that for any element $c \in R$ we have $a + b\lambda = uv$, where $uR + cR = R$, $vR + (1 - c)R = R$ and $uR + vR = R$. By Proposition 8.18 $\overline{R} = R/(a + b\lambda)R$ is an exchange ring. Since R is a commutative ring, by Proposition 2.17 it is a ring of idempotent stable range 1, i.e. stable range $\overline{R}$ equals 1. By Proposition 8.17 element $a + b\lambda$ is diadem. Theorem is proved. □

As the consequence of this theorem we have a next result.

Proposition 8.21. *Let R be a commutative Bezout ring of diadic range 1. Then for any ideal I of R factor-ring R/I is a ring of diadic range 1.*

Proof. Since homomorphic images of elementary divisor ring is elementary divisor ring, by Theorem 8.9 we have prove our proposition. □

Moreover, we have a next result

Proposition 8.22. *Let R is a commutative semi hereditary Bezout ring. If for any regular element (non zero divisor) $r \in R$ factor-ring R/rR is a ring of dyadic range 1 then R is a ring of dyadic range 1.*

Proof. By [**?**] if for a commutative semihereditary Bezout ring R a factor-ring R/rR for any regular element $r \in R$ is an elementary divisor ring, then R is an elementary divisor ring. By Theorem 8.9 a Proposition 8.22 is obvious

Consequently, we have that example of a commutative Hermite ring which is an elementary divisor ring [**?**] is example of a commutative Bezout ring of stable range 2 which is not of ring of diadic range 1.