Chapter 9
Problems related to range conditions

We provide a list of open problems that are connected to the commutative and non-commutative ring theory, K-theory and homological algebra. Some problems are solved now completely, some partially, and the most of them are without solutions, but are supplemented with ideas and hints.

The current paper contains a list of questions that were posed by the authors to the participants of the scientific seminar "The Problems of Elementary Divisor Rings" over the years. The formulated questions very from the simple checkup questions to the shorten versions of the further researches. Some questions are just the exercises, and some other are deep problems such that their solutions will be a nontrivial contribution to the general theory.

Eventually, the author would like to receive from the reader any information about the possible errors and inaccuracies in these questions. Moreover, it will be twice better to receive any ideas or even a complete solutions to the posed questions, but it is useful to remark that the answer to a question marked as "Open Problem" sometimes cannot be a sufficient reason of its publication.

One of the main sources of the almost all questions in the current paper is the rather old problem of the full description of the elementary divisor rings.

Moreover, it have been discovered that in the commutative rings case there exist structural theorems of these objects starting from some small values of stable range (for example, 1 or 2), that depends only on the considered problem, but not from the dimension or structure of the base ring.

Thus, for example, in the case of the commutative Bezout domains the elementary divisor rings are precisely the rings of neat range 1.

9.1 Rings ruled by annihilators and idempotents

The main attention here will be paid to the exchange, clean, regular, semiregular, unit-regular, coherent, P-injective, morphic, almost Baer rings. Although there are different ways to define these classes of ring, but they all have one common property: there are zero divisors inside them, and these divisors determine their structure. Each of the mentioned classes can be described using condition annihilators or/and idempotent. The latter features determine the title of the current section.

The adequate rings can be precisely described as rings whose finite homomorphic images are the semiregular rings. Also it is known that finite homomorphic images of commutative Bezout domain R are Kasch rings if and only if all maximal ideals of R are principal ones. Thus all finite homomorphic images of the adequate Bezout domains whose maximal ideals are principal are semiregular Kasch rings. Any commutative principal ideal domain can serve as an evident example.

Question 9.1. How can be described the rings whose finite homomorphic images are semiregular Kasch rings, but they are not commutative principal ideal domains? Can anyone find an example of such a ring?

Question 9.2. Describe the necessary and sufficient conditions for a P-injective ring being a semipotent one?

As the NJ-rings are a slight generalization of the von Neumann regular rings, the following group of questions is posted below.

Definition 9.1. A ring R is called an *NJ-ring* if every element $a \in R \setminus J(R)$ is a von Neumann regular element [**?**].

Question 9.3. Suppose that R is an NJ-ring, not necessarily commutative. If R is additionally a right Bezout ring of stable range 2 then is it a right Hermite one?

Question 9.4. If a NJ-ring R is a Hermite ring then is it true that R is an elementary divisor ring?

Definition 9.2. We say that R is a *separative ring* if the following condition holds for all finitely generated projective R-modules A and B:

$$A \oplus A \cong A \oplus B \cong B \oplus B \Rightarrow A \cong B.$$

Question 9.5. Is it true that the stable range of any separative NJ-ring equals to $1,2$ or infinity?

Question 9.5 is inspired by the result of Goodearl and Ara [**?**], who proved that stable range of the separative von Neumann regular ring equals to $1,2$ or ∞. We just want to know how the regularity condition can be weakened.

The next couple of problems are related to the full description of the Jacobson radical problem for the exchange ring. All these problems can be formulated for both commutative and noncommutative cases, but even the commutative one is not so easy.

Question 9.6. Suppose that R is a Bezout exchange ring. Is it a semiregular ring? Under what conditions it becomes a zero-adequate ring? If R is additionally a coherent / P-injective / morphic / IF ring, can we prove or disprove that it is a semiregular one? Are the inclusions

$$(Bezout\ morphic\ exchange) \subset (Bezout\ P-injective\ exchange) \subset$$

$$\subset (Bezout\ coherent\ exchange) \subset (Bezout\ exchange)$$

proper?

We finish this section with few problems concerning morphic rings.

Definition 9.3. A right (left) ideal I is called *right (left) pure* if the ring modulo this ideal is a flat left module.

Question 9.7. If R is a commutative morphic ring, then is it true that any finitely generated pure ideal is projective?

Question 9.8. What can one say about noncommutative morphic unit stable range 1 ring? Under what conditions a coherent / P-injective / morphic noncommutative ring R becomes unit stable range 1 ring?

The latter question is rather important, as any element of unit stable range 1 ring can be expressed as a sum of two units (these rings are called 2-*good* due to Vamos [**?**]). If we additionally suppose that any unit stable range 1 ring is a unit-central ring (a ring R is said to be *unit-central* if its units are central elements) then it becomes commutative.

9.2 Commutative Bezout Rings

The commutative Bezout rings are the natural generalization of principal ideal rings and they are not so "finite" as noetherian rings. This class of rings has not been so effectively studied as the principal ideal rings, noetherian, local and semilocal rings, but it has close relations with these classes as well as Bezout rings interpret GCD notion from integers to the abstract case.

If we suppose that R is a commutative Bezout domain then we obtain.

Question 9.9. Suppose that R is a commutative Bezout domain, and for any $a \in R \setminus \{0\}$ the matrix ring $M_n(R/aR)$ is semipotent. Is $M_n(R)$ a semipotent ring?

Question 9.10. Suppose that we have a commutative Bezout domain R and we consider the set S of all elements a such that any matrix $A \in M_2(R)$ with a at some position (i, j) admits canonical diagonal reduction. Is S a multiplicatively closed set? When the set S becomes a saturated one?

Question 9.11. Describe necessary and sufficient conditions for an element a of a commutative Bezout domain R such that $J(aR) = J(R/aR)$ is a noetherian module.

Question 9.12. It is known [**?**] that if any maximal ideal M of a ring R is pure, then R is a von Neumann regular ring. Therefore, what property is sufficient for maximal ideals of a commutative ring R to become an exchange ring?

9.3 Nontrivial Finite Homomorphic Images of Commutative Bezout Rings

There is the well know theorem of T.Shores [**?**] which states, that a commutative Bezout domain R is an elementary divisor ring if and only if so is any R/aR, where a is any nonzero element of R. Thus, a lot of properties of commutative Bezout rings will be studied using their finite homomorphic images R/aR.

Since, for a commutative Bezout domain R and any $a \in R \setminus \{0\}$ it is know that $Q_{cl}(R/aR) = R/aR$, studying an internal structure of R/aR is itself an interesting question.

Question 9.13. Suppose that R is a commutative Bezout domain, and $a \in R \setminus \{0\}$ is its fixed element.

1. How can one describe all $\overline{b} \in \overline{R} = R/aR$ such that if $\overline{b} \notin J(\overline{R})$ then $\overline{b}\overline{R}$ contains a nonzero idempotent? (i.e. when $\overline{R}$ is a semipotent ring?)

2. What are the elements $b \in R$ such that their images $\overline{b} \in \overline{R}$ are idempotents? a von Neumann regular elements?

Question 9.14. Let R be a commutative Bezout domain, and $a \in R \setminus \{0\}$. Is it true that R/aR is semipotent if and only if R is an almost stable range 1 ring?

Question 9.15. Suppose that R is a commutative Bezout ring, and $a \in R \setminus \{0\}$. Is it true that R/aR is a morphic ring in the case when R is a stable range 2 (or even almost stable range 1) ring?

Question 9.16. Suppose that R is a commutative Bezout domain, and $a = bc \in R \setminus \{0\}$, where $bR + cR = R$, $b \notin U(R)$, $c \notin U(R)$. Is R/aR an elementary divisor ring?

Definition 9.4. A commutative ring R is said to be an (f)-ring if any its pure ideal is generated by idempotents [**?**].

Question 9.17. Let R be a commutative Bezout domain and $a \in R \setminus \{0\}$. Under what conditions on the element a the ring R/aR is an (f)-ring?

The latter problem has a tight connection with adequate domains that will be stated in the next section.

Question 9.18. It is proved [**?**] that if the Jacobson radical $J(R)$ of a commutative coherent ring then is finitely generated or is flat then $R/J(R)$ is again a coherent ring. Using this fact for a commutative Bezout PM*-domain R and its element $a \in R \setminus \{0\}$ we need to know: if rad(R/aR) or $J(R/aR)$ is principal, then is a an adequate element?

9.4 Adequate Rings and Their Generalizations

We will consider some "locally" adequate ones.

Question 9.19. Suppose that for every nonzero element a of a commutative Bezout domain R and for every its nonzero ideal I there is some element $b \in I$ such that ${}_aA_b$. Is such R an elementary divisor ring?

Question 9.20. Suppose that R is a commutative Bezout domain and $a \in R \setminus \{0\}$. Is it true that ${}_aA_b$ if and only if $\overline{b}$ is an exchange element of the quotient ring R/aR?

Question 9.21. Let R be a commutative Bezout domain. Let us consider several axioms defined for elements of R.

1. For any triple of coprime elements $aR+bR+cR=R$ there is some $x\in R$ such that ${}_aA_{bx}$ or ${}_aA_{cx}$.

2. For any triple of coprime elements $aR+bR+cR=R$ we have that ${}_aA_b$ or ${}_aA_c$.

3. For any triple of coprime elements $aR+bR+cR=R$ there are some $x,y\in R$ such that ${}_{(a+cx)}A_{(b+cy)}$.

4. For any triple of coprime elements $aR+bR+cR=R$ there is some $p\in R$ such that $pR+bR+cR=R$ and ${}_cA_{ap}$.

5. For any triple of coprime elements $aR+bR+cR=R$ there are some $p,q\in R$ such that $pR+bR+cR=R$, $qR+aR+cR=R$, ${}_cA_{ap}$, ${}_cA_{bq}$.

6. Various modifications by increasing the number of coprime elements or adding x,y,p,q as factors in different combinations.

Therefore, the question is: is it true that any commutative Bezout domain admits some type of the adequacy (like some of axioms from 1 to 6)? Are there analogues with the stable range theory? Which of the axioms leads to well-known types of adequate rings? What are the examples of rings such that using them we can differ one of the written above axioms from other ones?

Definition 9.5. A ring R is called a von Neumann local (VNL) ring if either a either $1-a$ is a von Neumann regular element, for any $a\in R$ (see [**?**]).

Question 9.22. Let R be a commutative Bezout domain and $a,b,c\in R\setminus\{0\}$ are such that $aR+bR+cR=R$. Is it true that R/cR is a VNL-ring iff ${}_cA_a$ or ${}_cA_b$?

Question 9.23. Suppose that in every pair of coprime elements $a,b\in R\setminus\{0\}$ of a commutative Bezout domain R one of them is adequate. Is R/aR a VNL-ring for any $a\in R\setminus\{0\}$? Can the converse inclusion hold?

9.5 Local properties and ranges of commutative rings

As it was mentioned, a ring R is called a VNL-ring if for any $a\in R$ either a or $1-a$ is a von Neumann regular element. Here the von Neumann regularity holds only “locally” and from this fact implies their name. But what will happen if we change the von Neumann regularity by exchange, clean, semipotent, avoidable, adequate, stable range 1 or neat element property? What classes of rings will we obtain? Does some of these notions coincide or even are redundant?

Definition 9.6. We say that a ring R is a *locally P ring* if for any element $a \in R$ either a or $1-a$ satisfies the property P, where P is some ring-theoretical property that can be defined for a single element of a ring.

The concretization is, for example, the following:

Definition 9.7. An element a of a ring R is called a 2-simple element if there are some elements $u_1, u_2, v_1, v_2 \in R$ such that $u_1 a v_1 + u_2 a v_2 = 1$.

Definition 9.8. A ring R is called a *locally adequate (unit-regular,2-simple) ring* if for any element $a \in R$ either a or $1-a$ is an adequate (unit-regular, 2-simple) element.

This was a first point that we will discuss. The second one is a range property of a ring. The famous Dirichlet's theorem tells us that if a and b are coprime integers then in the arithmetical progression $\{a+bt\}$ there are infinitely many primes. We modify and extend this property in the following way.

Definition 9.9. A commutative ring R is of *P range* 1 if for any pair of coprime elements $a, b \in R$ there is some $t \in R$ such that element $a+bt$ satisfies the property P, where P is some ring-theoretical property that can be defined for a single element of a ring.

Definition 9.10. A commutative ring R is said to be a *ring of adequate range 1* if for any $a, b \in R$ such that $aR + bR = R$ there exist element $\lambda \in R$ such that $a + b\lambda$ is an adequate element of R. A commutative ring R is said to be a *ring of irreducible range 1* if for any $a, b \in R$ such that $aR + bR = R$ there exist element $\lambda \in R$ such that $a + b\lambda$ is an irreducible element of R. A commutative ring R is said to be a *ring of clean range 1* if for any $a, b \in R$ such that $aR + bR = R$ there exist element $\lambda \in R$ such that $a + b\lambda$ is a clean element of R.

It is well-known that if total ring of quotients $Q(R)$ of a commutative ring R is semilocal, then R has regular range 1, that is if $a, b \in R$ are regular elements (are not zero divisors) then there is some $t \in R$ such that $a + bt$ is again a regular element.

Thus we obtain several open problems.

Question 9.24. Is it true that R is a ring of neat range 1 if and only if for any element $c \in R$ and any coprime elements $a, b \in R$ there are some elements $x, y \in R$ such that $c = xy$ and $(1-ax)R + (1-by)R = R$?

Question 9.25. Does the condition $aR + bR = R$ implies that both elements are PM-elements, whether one of them is a PM-element? (A ring R is supposed to be commutative.)

Question 9.26. Does any locally adequate ring have an adequate range 1?

Now we will formulate several open problems connected the ones above but for a noncommutative case.

Question 9.27. Suppose that R is a noncommutative locally unit-regular ring. What is the stable range of R? If it is additionally an abelian ring then is R an exchange ring? A unit-regular ring?

Question 9.28. Is it true that any locally unit-regular Bezout ring is an elementary divisor ring if and only if it is an abelian ring (a ring R is called abelian if its idempotents are central)?

Question 9.29. When any simple locally 2-simple Bezout ring is an elementary divisor ring?

9.6 Noncommutative Bezout rings

It is proved in [**?**, **?**, **?**, **?**] that if R is a commutative Bezout domain and $a \in R \setminus \{0\}$ then the ring R/aR has several good properties: it is almost Baer, coherent, P-injective, morphic, IF-ring and $Q_{cl}(R/aR) = R/aR$. The following question naturally arises: what properties are preserved in the noncommutative case, especially when R is a duo-domain, or a is a duo-element of R?

Question 9.30. Suppose that R is a Bezout duo-domain and $a \in R \setminus \{0\}$. Has the ring R/aR one or two-sided properties from the next list: almost Baer property, IF-ring property, coherence, P-injectivity, morphic property? Does $Q_{cl}(R/aR) = R/aR$?

Question 9.31. Suppose that a is a duo-element of the noncommutative Bezout domain R. When R/aR is a PM-ring? P-injective ting? Morphic ring? Clean ring? Exchange ring? What can be said about the ring R or R/aR if R/aR is additionally a right Kasch ring?

B.V. Zabavsky and M.Ya. Komarnytskii in [**?**] have proved that the distributive Bezout domain is an elementary divisor ring if and only if it is a duo-ring. This result is quite good since being an elementary divisor ring requires some slight commutativity conditions in the case of distributive Bezout domain. On the other hand, the absence of zero divisors makes the theorem useful in a more narrow case than it is desired in general. Therefore some our hypothesis will be formulated in the following group of problems.

Question 9.32. Is it true that every quasi-duo stable range 1 Bezout ring is an elementary divisor ring if and only if it is a duo-ring?

Question 9.33. Let R be a quasi-duo Bezout ring such that $R/J(R)$ is an exchange ring. Is R an elementary divisor ring?

Question 9.34. Is a noncommutative semipotent stable range 1 Bezout ring an elementary divisor ring?

Question 9.35. Suppose that R is a noncommutative Bezout domain. Does it follow that R is an elementary divisor ring if so is $R/J(R)$? What about the case when $R/J(R)$ is a unit regular ring?

We will finish this section with a few commutativity problems for some special classes of rings.

Definition 9.11. A ring R is called an abelian (unit-central) ring if its idempotents (units) are central [**?**].

Question 9.36. It is known [**?**] that any unit-central ring R has stable range 1 if $R/J(R)$ is an exchange ring. Is every unit-central ring R a distributive ring if $R/J(R)$ is an exchange ring?

Definition 9.12. Let R be a commutative domain and there be a function $(N) : R \to \mathbb{Z}$ such that:

1. (N)(0)=0;

2. (N)(a)¿0 if $a \neq 0$.

Then we say that (N) is a norm on R. Element a of a commutative domain R with norm is said to be a ω-Euclidean element if for any nonzero element $b \in R$ we have a sequence of equations $a = bq_1 + r_1$, $b = r_1q_2 + r_2$, ... $r_{n-2} = r_{n-1}q_n + r_n$ where

$$(N)(r_n) < (N)(b)$$

A commutative domain is said to be a ring of ω-Euclidean range 1 if for any $a, b \in R$ such that $aR + bR = R$ there exist element $\lambda \in R$ such that $a + b\lambda$ is ω-Euclidean element and for any $\alpha, \beta \in R$ such that $\alpha R + \beta R = R$ we have $a + b\lambda = uv$ where $uR + \alpha R = R$, $vR + \beta R = R$ and $uR + vR = R$

Question 9.37. A Bezout domain is a domain in which any matrix can be reduced to the canonical diagonal form using only elementary column and row operations if and only if it is a domain of ω-Euclidean range 1.

9.7 Ultimate problems

We have called this section "Ultimate problems" since the solution of one of them will imply the solutions for many other problems in commutative (and noncommutative) ring theory as well as it will be a nontrivial contribution in the general algebraic investigations.

The first ultimate problem is the following one.

Problem 9.1. Is every right Bezout ring of stable range 2 a right Hermite one?

Sometimes the complete solution is too hard to find it at one time, therefore we have some other related questions.

Problem 9.2. Is every Bezout duo-ring of stable range 2 an Hermite ring?

Problem 9.3. Suppose that R is a commutative morphic ring. Is there any commutative Bezout domain K and some its element $a \in K \setminus \{0\}$ such that $R \cong K/aK$?

This means that the class of morphic rings coincides with the class of all nontrivial finite homomorphic images of the commutative Bezout domains.

Problem 9.4. Is every commutative Bezout domain an elementary divisor ring?